2025/2/28 Seminar @ Kanazawa university

Entanglement Rényi entropy and boson-fermion duality in massless Thirring model

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1. Main concepts of this talk

Quantum Entanglement in QFT

apply ←

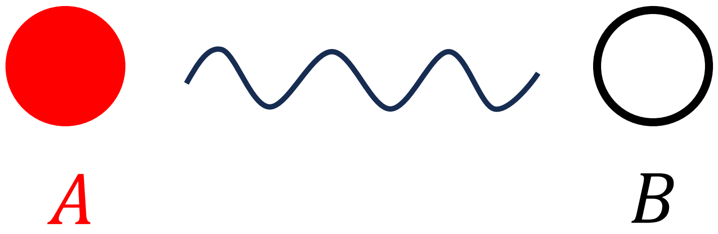
Boson-fermion duality

Often called **“bosonization”**

Take-home message: **Boson-fermion duality** can be used to analyze entanglement in **interacting** QFTs.

Entanglement = Correlations in quantum theory that cannot be explained by classical theory.

Example : two spin 1/2 system



Bell state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B\right) \xrightarrow{\text{Measurement}} \begin{cases} A = \uparrow & \Leftrightarrow \quad B = \uparrow \\ A = \downarrow & \Leftrightarrow \quad B = \downarrow \end{cases}$$

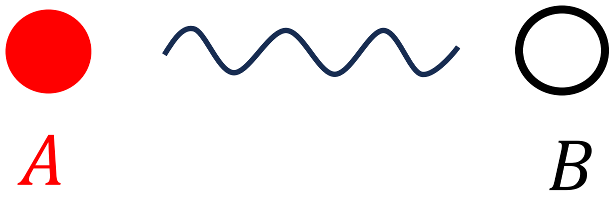
A and B are correlated through superposition

The notion of entanglement is important not only in quantum information theory but also in high-energy physics.

1. How can we quantify the entanglement?

Density matrix ： $\rho_{\mathrm{AB}} = |\psi_{AB}\rangle\langle\psi_{AB}|$

Reduced density matrix ： $\rho_A = \mathrm{Tr}_B[\rho_{\mathrm{AB}}]$



Entanglement Rényi Entropy (ERE) ：

$$S_n(A) \equiv \frac{1}{1-n} \log \mathrm{Tr}_A[\rho_{\mathrm{A}}^n] \quad , n \in \mathbb{Z}_+$$

$$\left(\lim_{n \to 1} S_n(A) = -\mathrm{Tr}_A[\rho_A \log \rho_A] \right)$$

Example : Bell state

Bell state ： $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B) \Rightarrow S_2(A) = -\log \mathrm{Tr}_A[\rho_{\mathrm{A}}^2] = \log 2 > 0$

Separable state (classical correlation): $|\psi'_{AB}\rangle = |\uparrow\rangle_A|\uparrow\rangle_B \Rightarrow S_2(A) = -\log \mathrm{Tr}_A[\rho_{\mathrm{A}}^2] = 0$

(we set $n = 2$ for simplicity)

ERE represents how much the two systems are quantumly entangled.

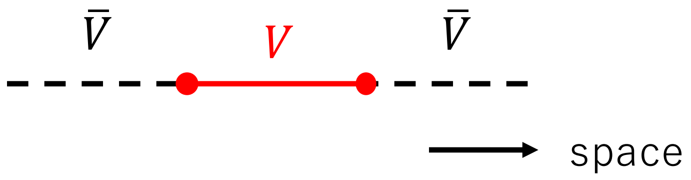
1. Quantum entanglement in QFT

In the case of QFT, there are degree of freedom on each special points.

system $A \longrightarrow$ region V

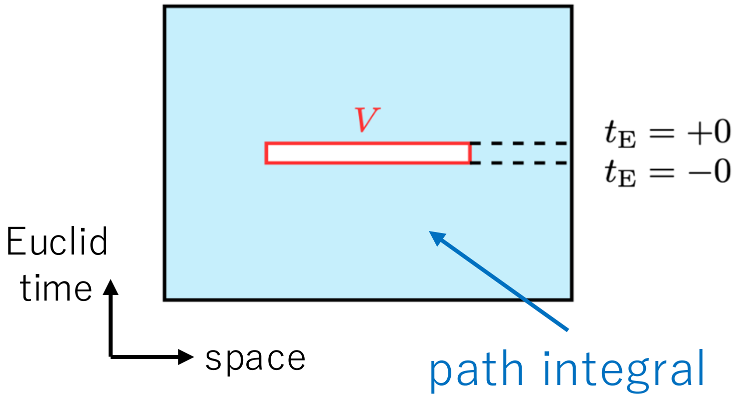
system $B \longrightarrow$ region $\bar{V}$ = complemental region of V

For (1+1)d



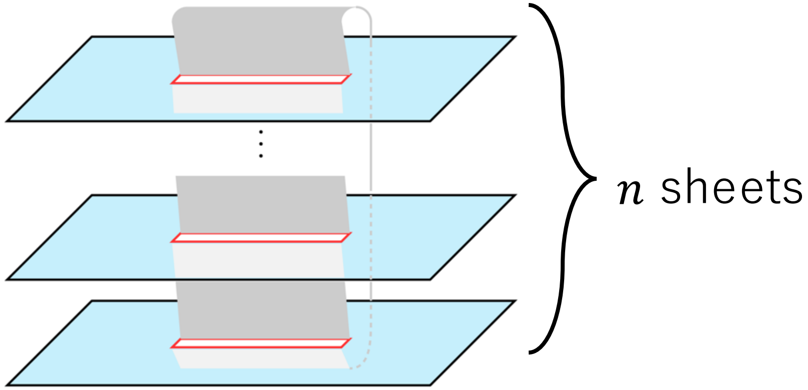
Replica method

$$\rho_V = \mathrm{Tr}_{\bar{V}}[|0\rangle\langle 0|]$$



Replicate

$$\mathrm{Tr}_V[\rho_V^n] \sim Z_n \quad \text{(partition function)}$$



n-sheeted manifold

The ERE reduces to the partition function on the n-sheeted manifold.

1. Quantum entanglement in QFT

The replica method works well for CFTs and free theories.

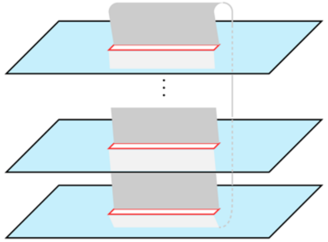
Previous works:

For (1+1)d,

- ✔ CFT, V = single interval. [Holzhey,Larsen, Wilczek 1994]
- ✔ massless free fermion, $V = N$-intervals. [Casini, Fosco, Huerta 2005]
- ✔ massless free boson, V =two-intervals. [Calabrase, Cardy, Tonni 2011]

In those cases, we can derive the exact result of ERE.

However, the calculation of entanglement is very difficult for interacting theories⋯



→ difficult to calculate⋯

There are almost no examples of rigorous analytical calculations of the effects of interactions on entanglement in QFT.

Our aim : To precisely understand how interactions contribute to entanglement in QFT.

Our idea : Boson-fermion duality [Karch, Tong, Turner 2019]

Fermionic theory		Bosonic theory
Local operator : $\hat{O}^F$	ferminization ←	Local operator : $\hat{O}^B$
Discrete symmetry : $\mathbb{Z}_2^F$	→ bosonization	Discrete symmetry : $\mathbb{Z}_2^B$
Partition function : Z^F		Partition function : Z^B

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There is a correspondence between partition functions

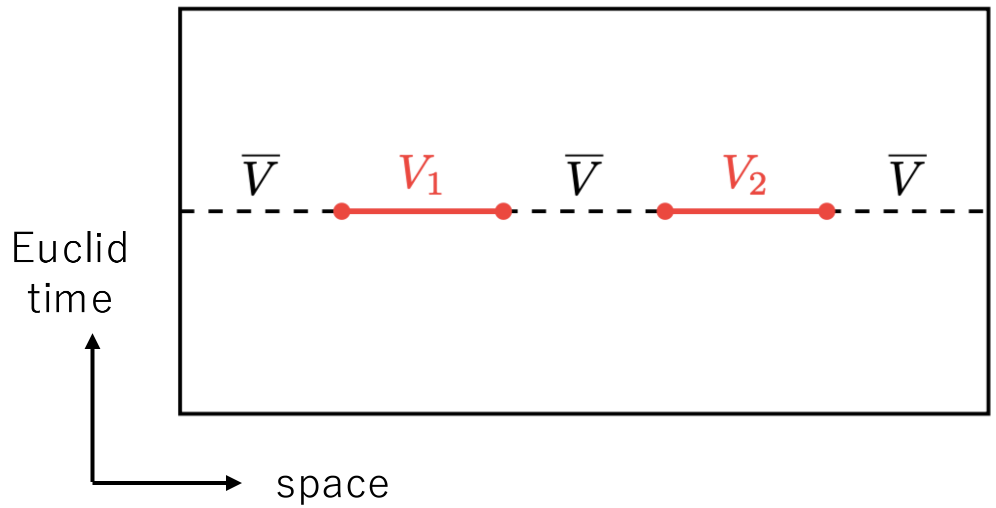
1. Short summary of our work

What we did：

- By combining the replica method and boson-fermion duality, we perform rigorous analytical calculations of the entanglement Rényi entropy (ERE) in interacting models.
- The model is the massless Thirring model (1+1 dimensions, fermion with a four-points interaction).
- We set $V = V_1 \cup V_2$ (two intervals), which allows us to observe the effect of interaction.
- Exact results reveal the non-perturbative behavior of the ERE.

massless Thirring model [Thirring 1958]

$$\mathcal{L}_F = i\,\bar{\psi}\,\gamma^\mu \partial_\mu \psi + \frac{\pi}{2}\lambda \underbrace{(\bar{\psi}\,\gamma^\mu \psi)(\bar{\psi}\,\gamma_\mu \psi)}_{\text{interaction}}$$



Outline

1. Introduction
2. Boson-fermion duality
3. Entanglement in massless Thirring model
4. Results
5. Summary and future works

Outline

2. Boson-fermion duality

In (1+1) dimension, certain fermionic theories can correspond to bosonic theories.

Fermionic theory		**Bosonic theory**
Local operator : $\hat{O}^F$	ferminization ←	Local operator : $\hat{O}^B$
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2. Boson-fermion duality

In (1+1) dimension, certain fermionic theories can correspond to bosonic theories.

Fermionic theory

Local operator : $\hat{O}^F$

Discrete symmetry : $\mathbb{Z}_2^F$

Partition function : Z^F

ferminization

bosonization

Bosonic theory

Local operator : $\hat{O}^B$

Discrete symmetry : $\mathbb{Z}_2^B$

Partition function : Z^B

In this talk, I will explain how a fermionic theory can be constructed from a bosonic theory.

An Example of boson-fermion duality

Let us consider a two-dimensional Euclidean spacetime with coordinates (x, τ_{E}).

$z = x + i\tau_{\mathrm{E}}$: complex coordinate

$\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$

Massless free fermion

$$\mathcal{L}_F = \frac{1}{2\pi}\left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right)$$

$\psi(z), \bar{\psi}(\bar{z})$: left(right) moving fermion

$$\langle \psi(z)\psi^{\dagger}(w) \rangle = \frac{1}{z - w}$$

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Massless free compact boson

$$\mathcal{L}_B = \frac{1}{2\pi}\partial\phi\bar{\partial}\phi$$

$\phi \sim \phi + 2\pi$: compact boson

$$\phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$$

$$\langle\varphi(z)\varphi(w)\rangle = -\ln(z - w)$$

Vertex operator : $V(z) \equiv :e^{i\varphi(z)}:$

$$\left\langle V(z)V^{\dagger}(w)\right\rangle = e^{\langle\varphi(z)\varphi(w)\rangle} = \frac{1}{z - w}$$

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Vertex operator : $V(z) \equiv :e^{i\varphi(z)}:$

$$\boxed{\langle V(z)V^\dagger(w)\rangle = \frac{1}{z-w}}$$

Correspondence

$$\psi(z) \longleftrightarrow V(z) \equiv :e^{i\varphi(z)}:$$

$$\mathbb{Z}_2^F : \psi(z) \to -\psi(z) \longleftrightarrow \mathbb{Z}_2^B : \varphi(z) \to \varphi(z) + \pi$$

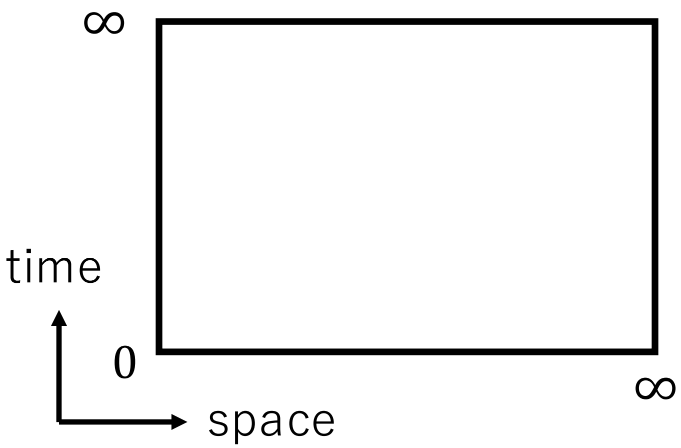
Boson-fermion duality

The fermionic theory $\mathcal{T}_F$ and the bosonic theory $\mathcal{T}_B$ are related to each other through "boson-fermion duality".

$$\overset{def}{\Longleftrightarrow} \left\{ \begin{array}{l} \text{There exists a fermionic operator } \mathcal{O}_F \text{ defined in } \mathcal{T}_F \text{ and a corresponding} \\ \text{bosonic operator } \mathcal{O}_B \text{ defined in } \mathcal{T}_B\text{:} \\ \qquad\qquad\qquad \mathcal{O}_F \leftrightarrow \mathcal{O}_B \\ \text{And the observables of the fermionic theory coincide with those of the} \\ \text{bosonic theory:} \\ \qquad \langle \mathcal{O}_F(x)\mathcal{O}_F(y)\cdots\rangle = \langle \mathcal{O}_B(x)\mathcal{O}_B(y)\cdots\rangle \end{array} \right.$$

2. Boson-fermion duality

How do partition functions correspond ?



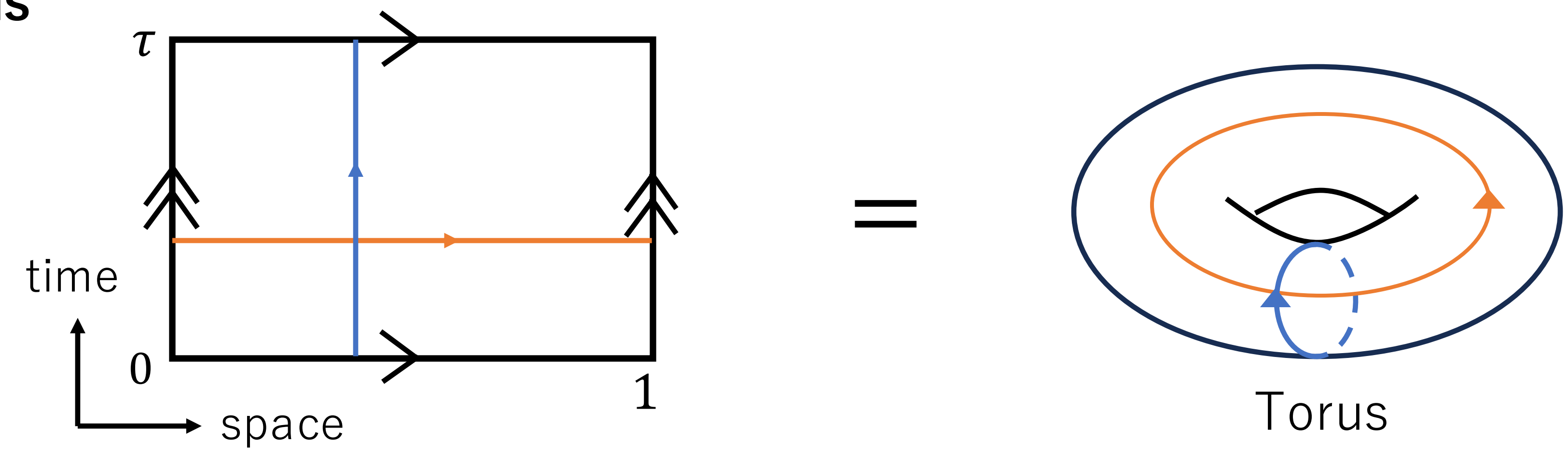
The correspondence of partition functions on plane $\mathbb{R}^2$:

$$Z_F = Z_B$$

The partition functions of the fermionic theory and the bosonic theory coincide without subtlety.

However, a subtlety arises in spacetimes with non-trivial topology, such as that of a torus.

Example: Torus



Spin structure ϱ

↔ Periodicity of the fermion field along the cycles.

For torus,

$$\begin{cases} \psi(z+1) = \pm\psi(z) \\ \psi(z+\tau) = \pm\psi(z) \end{cases}$$

P : Periodic
A : Anti-periodic

$$\rightarrow \varrho = \mathrm{AA}, \mathrm{AP}, \mathrm{PA}, \mathrm{PP}$$

$Z_F[\varrho]$ does depend on ϱ

Z_B does not depend on ϱ

How do partition functions correspond?

2. Boson-fermion duality

How can we construct the fermionic theory in spacetimes with topology from the bosonic one ?

This question is answered as follows: [Karch-Tong-Turner2019]

STEP 1 : Couple the bosonic theory $\mathcal{T}_B$ with the topological QFT called Kitaev.

$$\mathcal{T}_B \times \underbrace{\mathrm{Kitaev}}_{\text{Depends on the spin structure } \varrho}$$

STEP 2 : Gauging the $\mathbb{Z}_2^B$ global symmetry.

$$\frac{\mathcal{T}_B \times \mathrm{Kitaev}}{\mathbb{Z}_2^B} = \underbrace{\mathcal{T}_F}_{\text{Get the fermionic theory properly.}}$$

STEP 1 : Kitaev

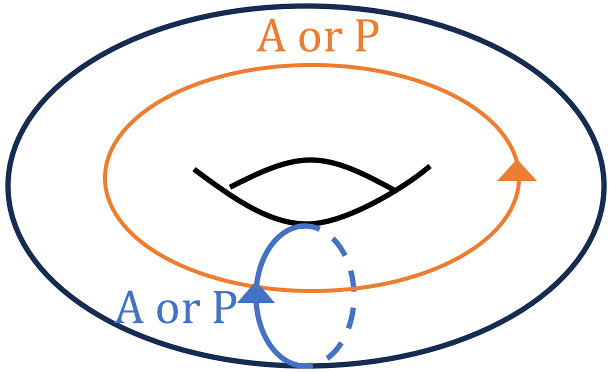
The partition function of the fermionic spin chain Kitaev is given by:

$$Z_{\mathrm{Kitaev}} = (-1)^{\boxed{Arf[\varrho]}+Arf[T+\varrho]}$$

$Arf[\varrho] \in \{0,1\}$: Arf invariant

T : background $\mathbb{Z}_2^B$ gauge field

Example : Torus



$$Arf[\varrho] = \begin{cases} 0\,, & \varrho = \mathrm{AA}, \mathrm{AP}, \mathrm{PA} \\ 1\,, & \varrho = \mathrm{PP} \end{cases}$$

※For (1+1)d, $Arf[\varrho]$ can be calculated using the mod 2 index.

STEP 1 : Kitaev

The partition function of the fermionic spin chain Kitaev is given by:

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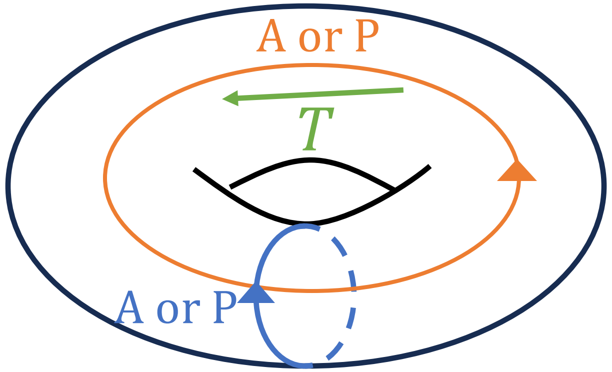
Example : Torus



The presence of the background $\mathbb{Z}_2^B$ gauge field changes the spin structure:

$$\varrho \to \varrho_T = T + \varrho$$

Let me explain the meaning on the next slide.

$\mathbb{Z}_2^B$ gauge field

Mathematically, the $\mathbb{Z}_2^B$ gauge field is given as an element of cohomology : $T \in H^1(X, \mathbb{Z}_2^B)$

Instead of a mathematical definition, I will provide a physical intuition.

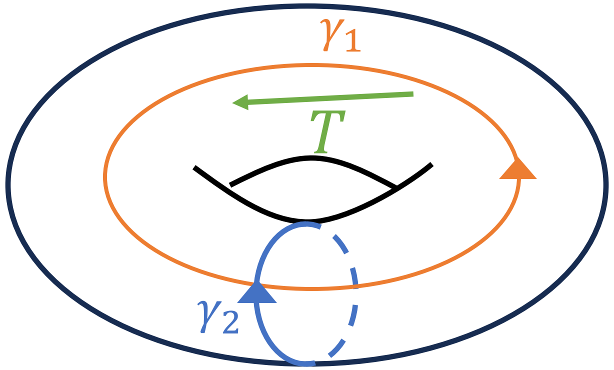
X : spacetime manifold

We characterize the $\mathbb{Z}_2^B$ gauge field by its holonomy:

$$T = (T_1, T_2)\,, \qquad T_I = \oint_{\gamma_I} T \in \underbrace{\{0,1\}}_{\mathbb{Z}_2 \text{ valued}}$$

There are four configurations of the $\mathbb{Z}_2^B$ gauge field:

$$T = (0,0), (0,1), (1,0), (1,1)$$



Example : torus

The change in the spin structure: (Aharanov-Bohm effect)

$T = (1,0)$ and $\varrho = \mathrm{AA} \Rightarrow T + \varrho = \mathrm{PA}$

$T = (0,1)$ and $\varrho = \mathrm{AA} \Rightarrow T + \varrho = \mathrm{AP}$

$\vdots$

The result of STEP 1 :

$$\mathcal{T}_B \times \text{Kitaev}: \; Z_B[T]\underbrace{(-1)^{Arf[\varrho]+Arf[T+\varrho]}}_{Z_{\text{Kitaev}}}$$

ϱ : spin structure of spacetime X

$Arf[\varrho] \in \{0,1\}$: Arf invariant

$T \in H^1(X, \mathbb{Z}_2^B)$: $\mathbb{Z}_2^B$ background gauge field

$Z_B[T]$: partition function of the bosonic theory with a background $\mathbb{Z}_2^B$ gauge field T

STEP 2 : Gauging the $\mathbb{Z}_2^B$ global symmetry.

Promote the **background** gauge field T to a **dynamical** one t .

=**Some fixed value**

=**Sum over all configurations of the gauge field.**

$$\frac{\mathcal{T}_B \ \times \ \text{Kitaev}}{\mathbb{Z}_2^B} \quad : \quad Z_X^F[\varrho] = \frac{1}{2^g} \sum_{t \in H^1(X, \mathbb{Z}_2^B)} Z_X^B\,[t](-1)^{Arf[\varrho]+Arf[t+\varrho]}$$

X : spacetime manifold, g : the number of genus

t : dynamical $\mathbb{Z}_2^B$ gauge field.

STEP1 and STEP2 give the partition function of the dual fermionic theory.

Fermionization dictionary

$$\boxed{Z_X^F[\varrho]} = \frac{1}{2^g} \sum_{t \in H^1(X, \mathbb{Z}_2^B)} \boxed{Z_X^B[t]} \, (-1)^{Arf[\varrho] + Arf[t+\varrho]}$$

fermion　　boson

X : spacetime manifold

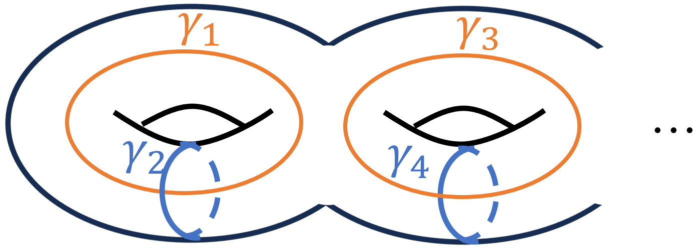
g : the number of genus

ϱ : spin structure

t : $\mathbb{Z}_2^B$ gauge field

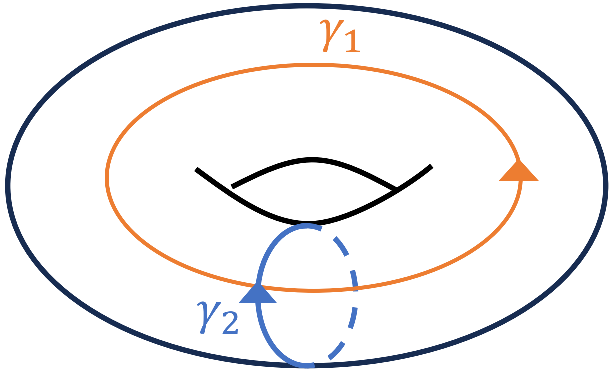
※ This fermionization dictionary holds for general two-dimensional Rieman surfaces, not just the torus.

The order of $H^1(X, \mathbb{Z}_2^B)$ is 2^{2g}.



2. Boson-fermion duality

Fermionization dictionary for torus $(g = 1)$



$$Arf[\varrho] = \begin{cases} 0\,, \varrho = \mathrm{AA}, \mathrm{AP}, \mathrm{PA} \\ 1\,, \varrho = \mathrm{PP} \end{cases}$$

The $\mathbb{Z}_2^B$ gauge field takes 4 configurations:

$$t = (0{,}0), (0{,}1), (1{,}0), (1{,}1)$$

Fermionization dictionary

$$Z_{\mathbf{T}}^F[\mathrm{AA}] = \frac{1}{2}\left(Z_{\mathbf{T}}^B[00] + Z_{\mathbf{T}}^B[01] + Z_{\mathbf{T}}^B[10] - Z_{\mathbf{T}}^B[11]\right)$$

fermion boson

If we know all the partition functions of the bosonic theory $Z_{\mathbf{T}}^B[t]$, we can derive the partition functions of the dual fermionic theory $Z_{\mathbf{T}}^F[\varrho]$.

Outline

1. Introduction
2. Boson-fermion duality
3. Entanglement in massless Thirring model
4. Results
5. Summary and future works

2. Entanglement in massless Thirring model

In our study, we consider the following fermionic theory as an interacting theory.

massless Thirring model

$$\mathcal{L}_F = \underbrace{i\,\bar{\psi}\,\gamma^\mu \partial_\mu \psi}_{\text{Free Dirac}} + \frac{\pi}{2}\lambda\,\underbrace{(\bar{\psi}\,\gamma^\mu \psi)(\bar{\psi}\,\gamma_\mu \psi)}_{\text{4-point interaction}}$$

ψ : Dirac fermion

λ : Thirring coupling

Remarks :

1. This model was introduced by W.E. Thirring as a solvable quantum field theory in (1+1) d. [W.E. Thirring 1958]
2. This model is a marginal deformation of the free Dirac fermion.

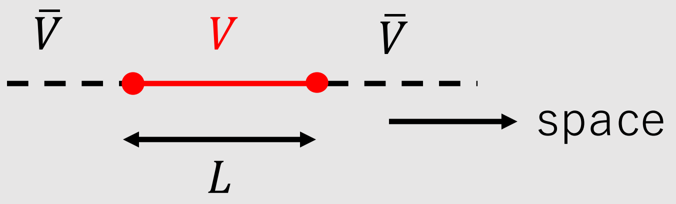
   → The massless Thirring model still possesses conformal symmetry.
3. The dual bosonic theory is a free compact boson. [S.R. Coleman 1975]

2. Entanglement in massless Thirring model

One of the previous results

CFT, Single interval : $V = [0, L]$

[Holzhey,Larsen, Wilczek 1994]



For a general CFT, the EE is given by:

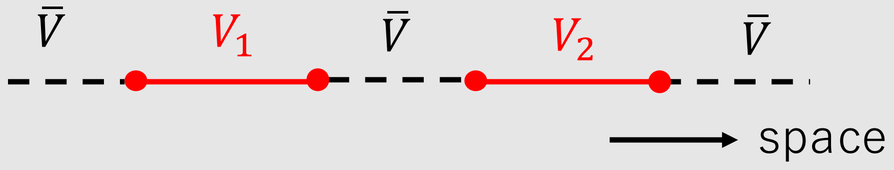
$$S_V = \frac{c}{3} \log \frac{L}{\epsilon}$$

c : central charge of a CFT
ϵ : UV cutoff scale

We cannot observe the coupling dependence.

Our setting

2-interval : $V = V_1 \cup V_2$



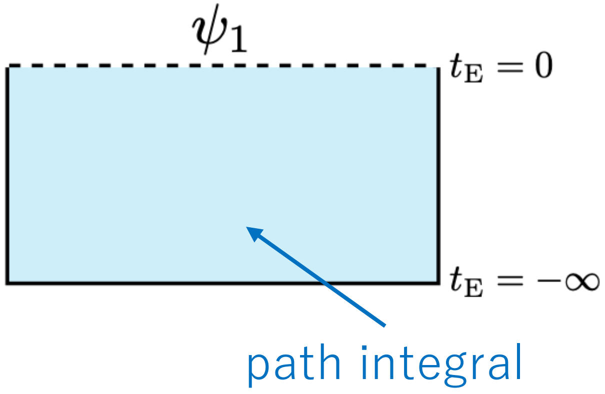
In our case, the EE is not determined solely by the central charge:

$$S_V = S_V(\boxed{\lambda}, V_1, V_2, c, \epsilon)$$

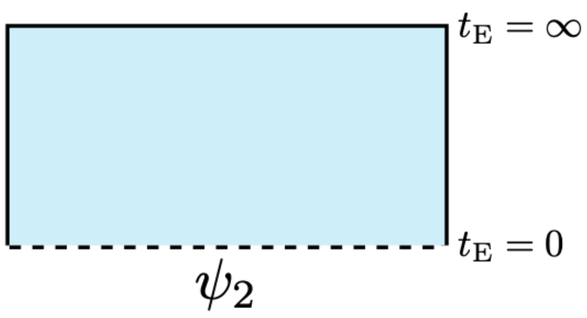
We can see how the interaction contributes to the entanglement.

Replica method

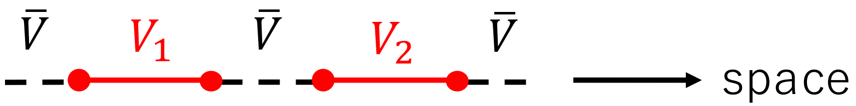
$\langle \psi_1 | 0 \rangle =$



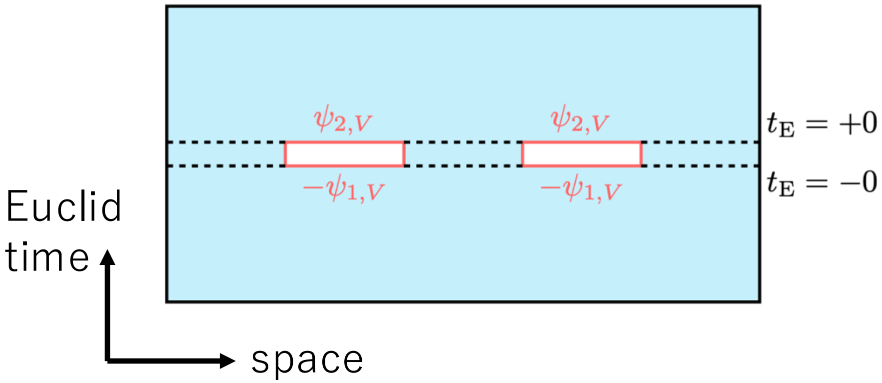
$\langle 0 | \psi_2 \rangle =$



$V = V_1 \cup V_2$



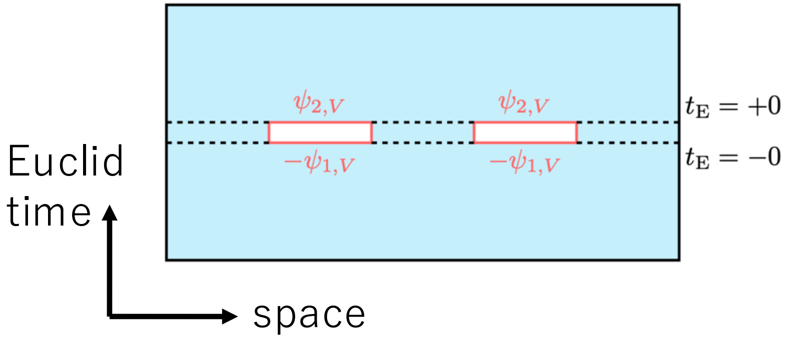
$$\rho_V(\psi_1, \psi_2) = \mathrm{Tr}_{\bar{V}}[\langle \psi_1 | 0 \rangle \langle 0 | \psi_2 \rangle] =$$



Replica method

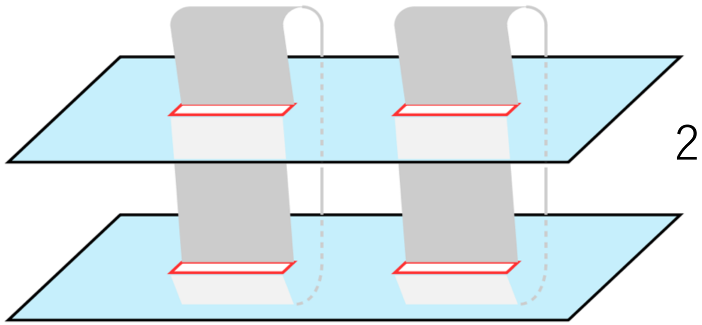
For simplicity, we consider the second Rényi entropy $S_2(V) = -\log \mathrm{Tr}_V[\rho_V^2]$

$$\rho_V(\psi_1, \psi_2) =$$



$$\mathrm{Tr}_V[\rho_V^2] = \sum_{\psi_1,\psi_2} \rho_V(\psi_1,\psi_2)\, \rho_V(\psi_2, -\psi_1) \sim Z^F_{\Sigma_{2,2}} =$$

gluing



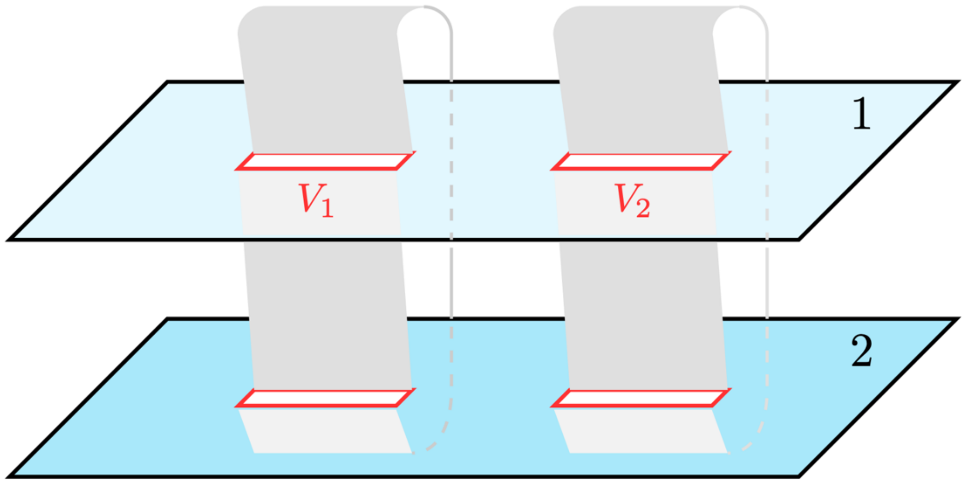
2-sheeted manifold

$\Sigma_{2,2}$

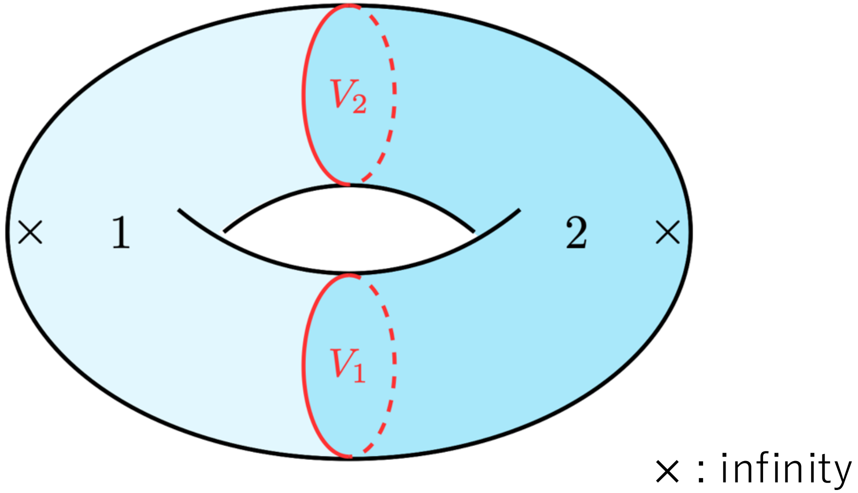
We have to calculate the partition function on $\Sigma_{2,2}$

2. Entanglement in massless Thirring model

$\Sigma_{2,2}$ can be mapped to a torus by the conformal map. [Lunin, Mathur 2001]



2-sheeted manifold $\Sigma_{2,2}$



Torus $\mathbf{T}$

cross-ratio : $x = \dfrac{(v_1 - u_1)(v_2 - u_2)}{(u_2 - u_1)(v_2 - v_1)}$ ⟷ modulus : τ

⇒ $Z^F_{\Sigma_{2,2}} \sim Z^F_{\mathbf{T}}$ The ERE reduces to partition function on a torus.

2. Entanglement in massless Thirring model

The way to calculate the partition function on the torus $Z_{\mathbf{T}}^{F}$ is through boson-fermion duality.

massless Thirring model

$$\mathcal{L}_F = i\,\bar{\psi}\,\gamma^{\mu}\partial_{\mu}\psi \;+\frac{\pi}{2}\lambda\,\underbrace{(\bar{\psi}\,\gamma^{\mu}\psi)(\bar{\psi}\,\gamma_{\mu}\psi)}_{\text{interaction}}$$

ψ : Dirac fermion

λ : Thirring coupling

$\mathbb{Z}_2^F$: $\psi \to -\psi$

It is difficult to analyze due to the interaction.

fermionization

$$1+\lambda = \frac{4}{R^2}$$

free compact boson

$$\mathcal{L}_B = \frac{R^2}{8\pi}\,\partial_{\mu}\phi\,\partial^{\mu}\phi$$

$$\phi \sim \phi + 2\pi$$

ϕ : scalar field

R : compact boson radius

$\mathbb{Z}_2^B$: $\phi \to \phi + \pi$

It is easy to analyze.

We can obtain the partition function $Z_{\mathbf{T}}^{F}$ from the bosonic side.

2. Entanglement in massless Thirring model

The flow of our analysis :

$$S_2(V) \overset{\text{Replica method}}{=} -\log Z^F_{\Sigma_{2,2}}$$

Conformal map

$$= -\log\left(f(V) \times Z^F_{\mathbf{T}}\right)$$

$f(V)$: Conformal factor

Boson-fermion duality

$$= -\log\left(f(V) \times \frac{1}{2}\underbrace{\left[Z^B_{\mathbf{T}}[00] + Z^B_{\mathbf{T}}[01] + Z^B_{\mathbf{T}}[10] - Z^B_{\mathbf{T}}[11]\right]}_{\text{Partition functions of the free theory}}\right)$$

We reduced the calculation of the Rényi entropy in massless Thirring model to that of the partition functions of the free bosonic theory.

Outline

Our analytical result

[**H. F**, T. Nishioka, S. Shimamori, 2023]

$$S_2(V,\lambda) = S_2(V,0) - \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2(\tau(1+\lambda))\,\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]$$

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V
τ : moduli of torus
λ : coupling const
$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

Our analytical result

[**H. F**, T. Nishioka, S. Shimamori, 2023]

$$S_2(V,\lambda) = \boxed{S_2(V,0)} - \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2(\tau(1+\lambda))\,\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]$$

Free term

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V
τ : moduli of torus
λ : coupling const
$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

Consistent with previous result (free fermion).

3. Results

Our analytical result

[**H. F**, T. Nishioka, S. Shimamori, 2023]

$$S_2(V,\lambda) = S_2(V,0) \boxed{- \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2(\tau(1+\lambda))\,\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]}$$

interaction term

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V
τ : moduli of torus
λ : coupling const
$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

For $\lambda = 0$, this term vanishes from Jacobi identity $\vartheta_3^4(\tau) - \vartheta_2^4(\tau) - \vartheta_4^4(\tau) = 0$

Our analytical result

[**H. F**, T. Nishioka, S. Shimamori, 2023]

$$S_2(V,\lambda) = S_2(V,0) \boxed{- \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2(\tau(1+\lambda))\,\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]}$$

interaction term

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V

τ : moduli of torus

λ : coupling const

$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

Arbitrary λ

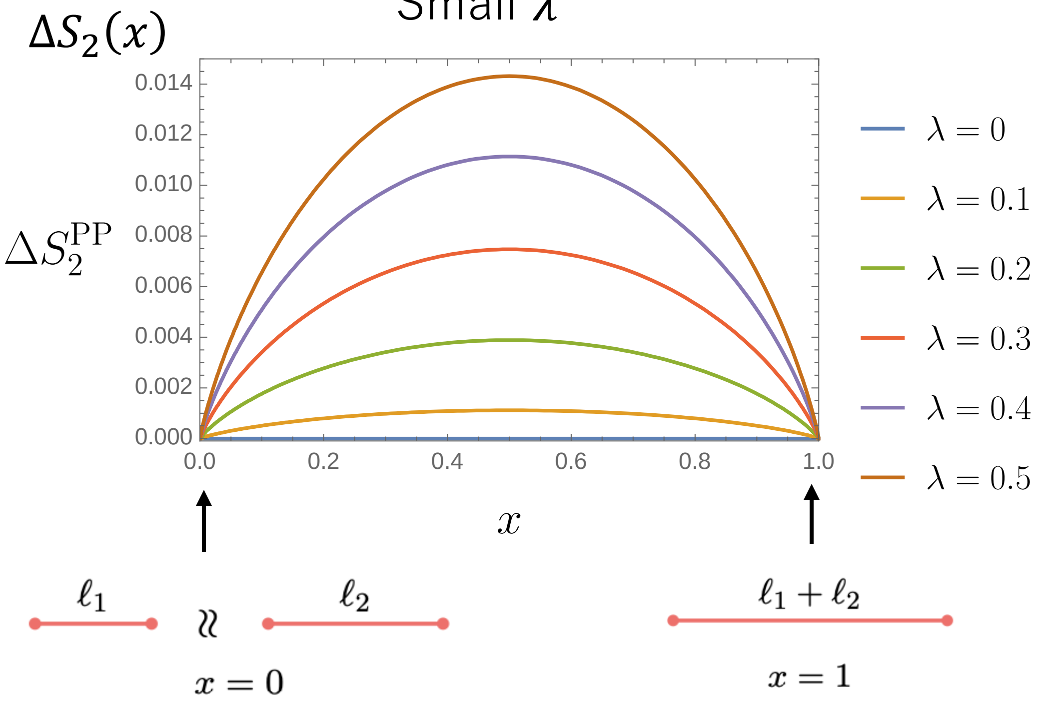
➡ **We derived the Rényi entropy for an interacting QFT exactly.**

3. Results

Let us examine the entangling region dependence of the entanglement :

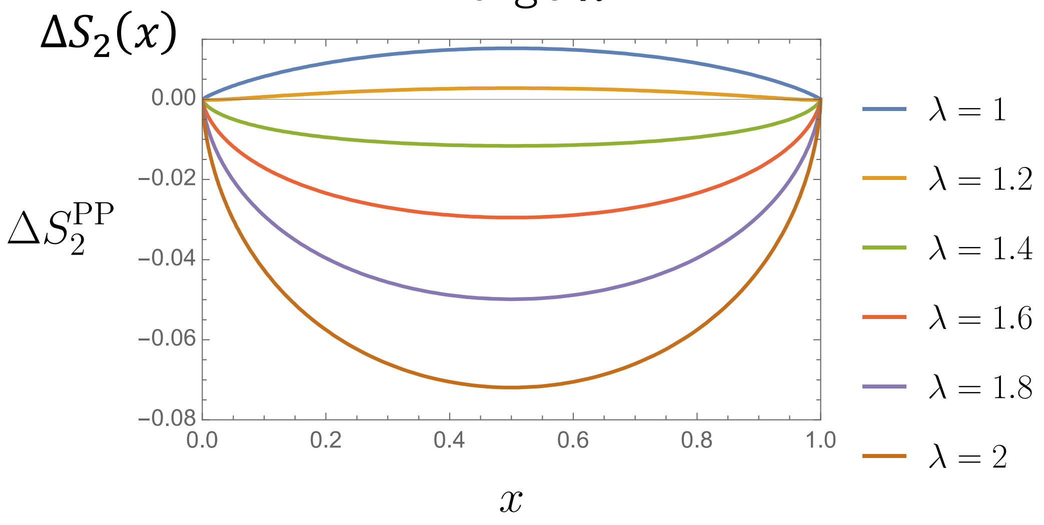
$$\Delta S_2(x) = S_2(V,\lambda) - S_2(V,0)$$

Small λ



x : cross-ratio of region V

Large λ



CFT, V =1-interval [Holzhey et al 1994]

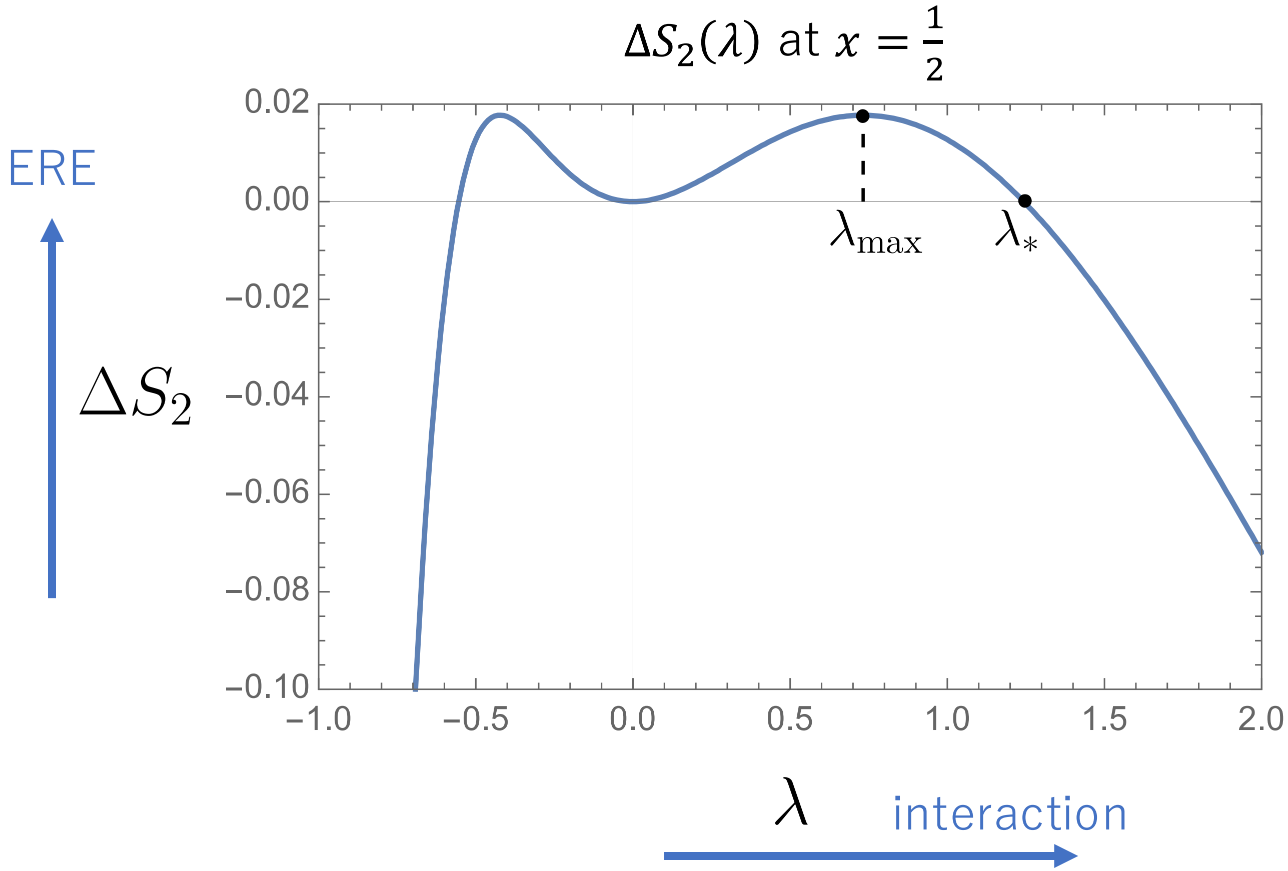
$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left(\frac{v-u}{\epsilon}\right)$$

c : central charge,
ϵ : UV cutoff

Our result is consistent with previous work and new results

3. Results

Let us examine the interaction dependence : $\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$



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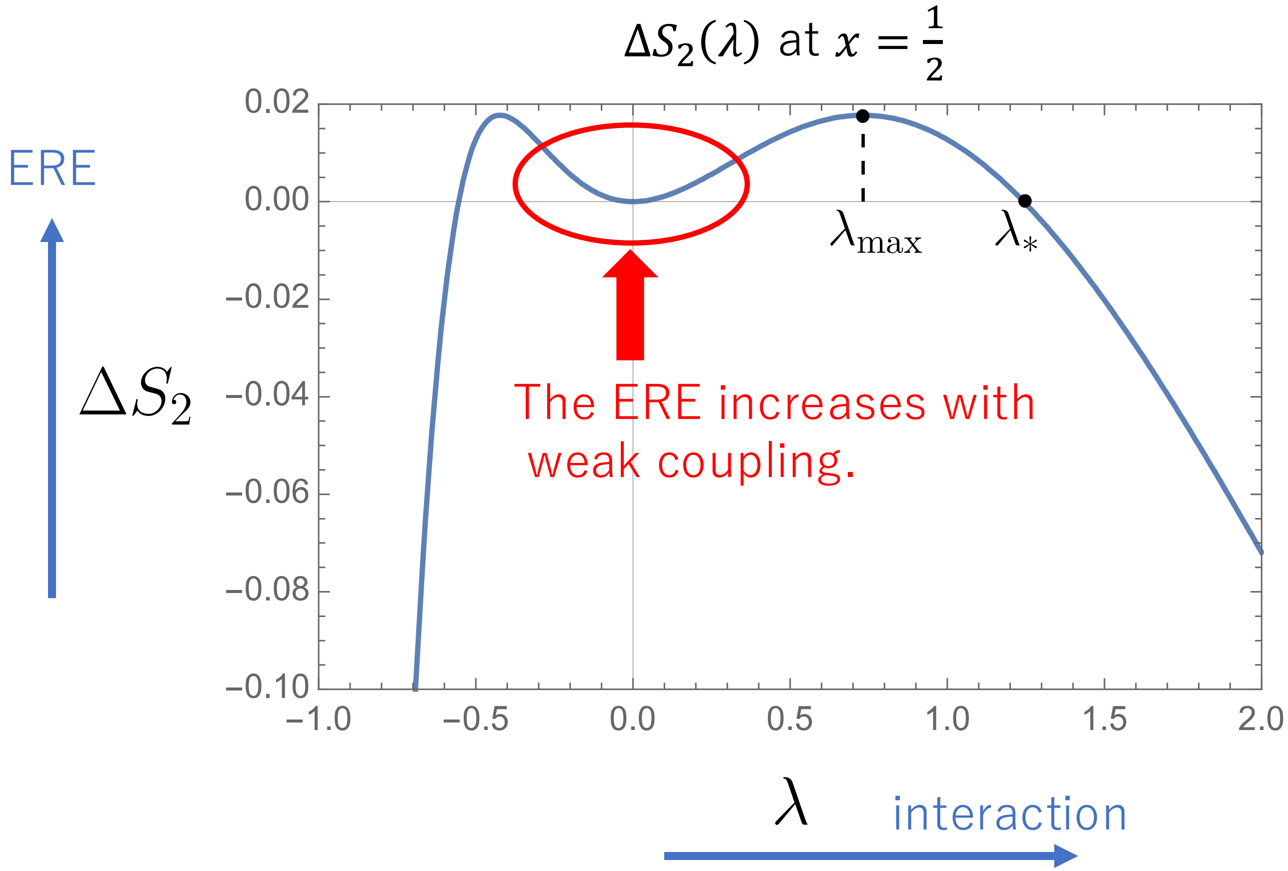
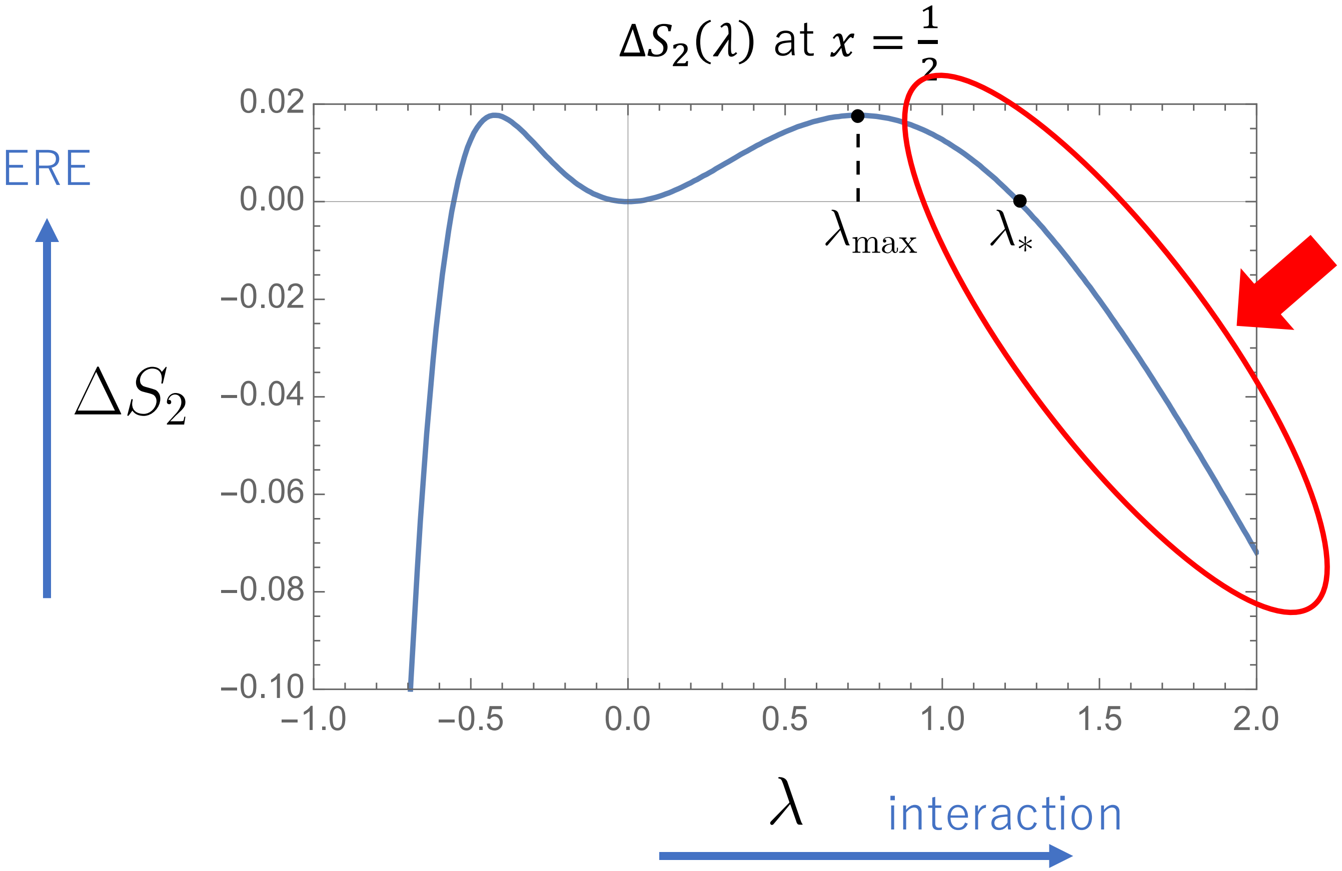


3. Results

Let us examine the interaction dependence : $\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$

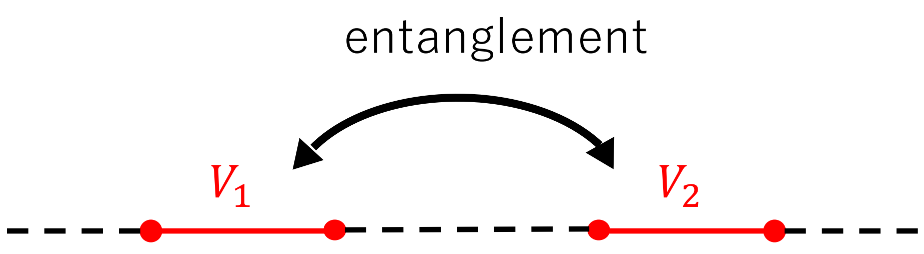


Unlike the perturbative regime, the ERE decreases in strong coupling regime.

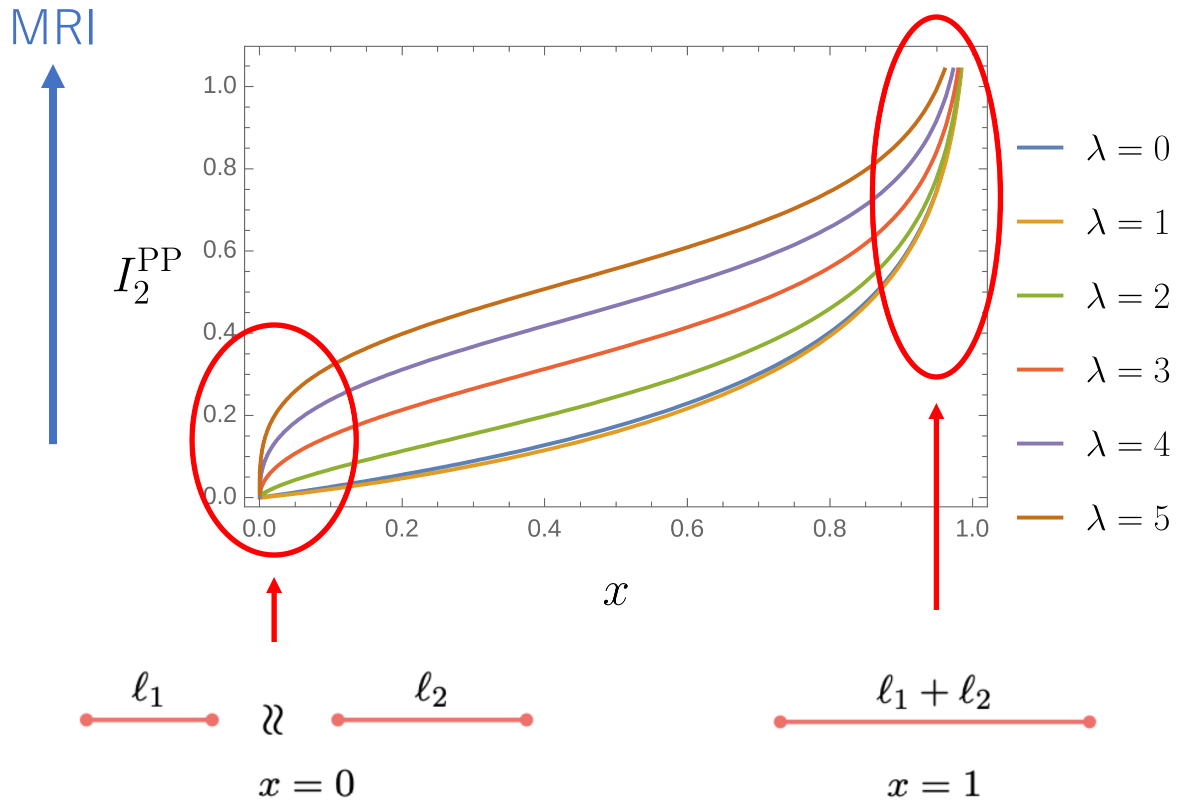
We explore the interaction dependence of the ERE, including the non-perturbative regime.

3. Results

Mutual Rényi information (MRI) : $I_n(V_1, V_2) = S_n(V_1) + S_n(V_2) - S_n(V_1 \cup V_2)$



- $x \sim 0, \quad x \sim 1$: reasonable behavior



3. Results

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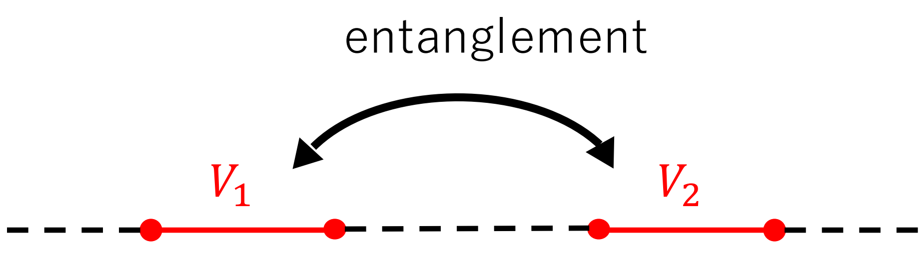
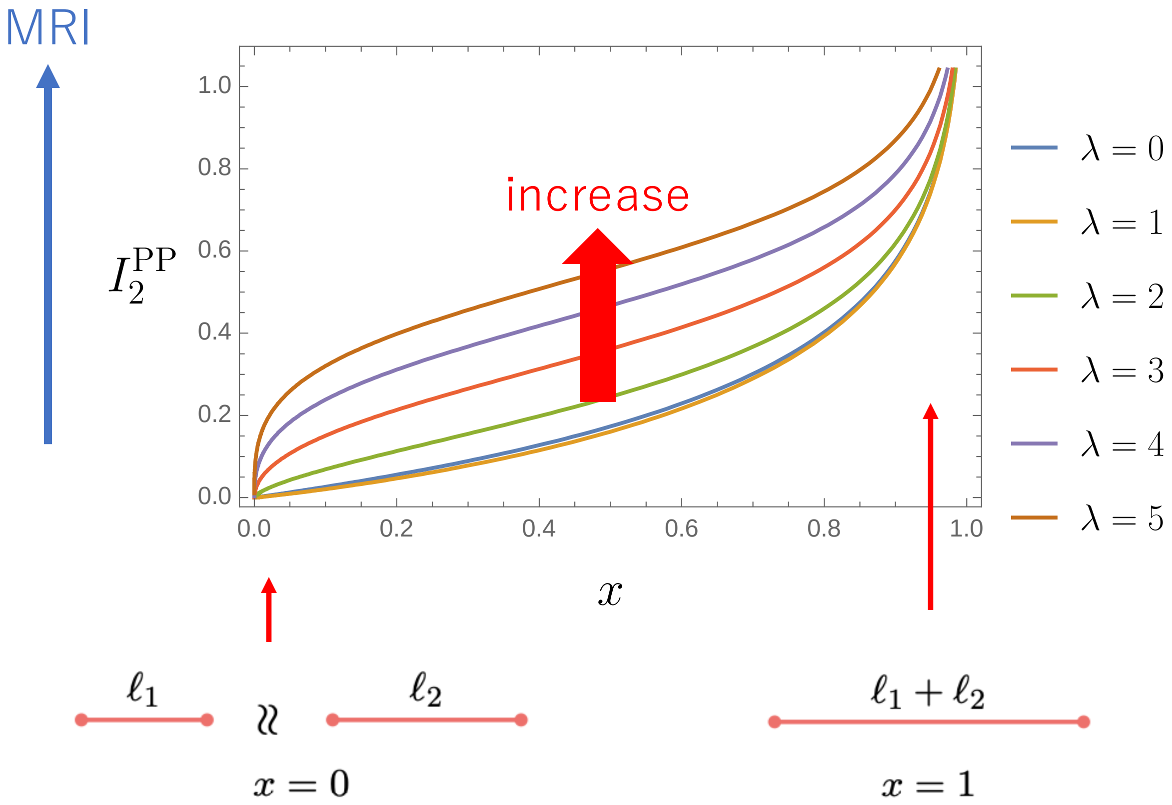


- $x \sim 0$, $x \sim 1$: reasonable behavior
- MRI increase as the coupling const increase.



Outline

4. Summary and future works

Summary

- Entanglement is an important concept not only in quantum information theory but also in high-energy physics.
- However, calculating the effect of interaction in QFT is a difficult task.
- We combined the replica method with boson-fermion duality to address this issue.
- We derived the ERE and MRI exactly in an interacting system and investigated the entanglement, including the non-perturbative regime.

Comment on subsequent research [Marić, Bocini, Fagotti,2023]

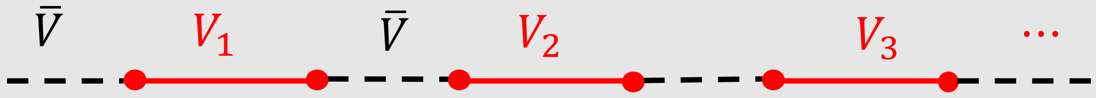
They explore the ERE in XXZ spin chain (↔ massless Thirring model)

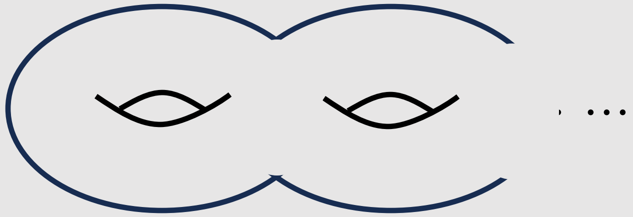
➡ Their results are consistent with ours.

4. Summary and future works

Future direction

- Increasing the number of intervals or replica sheets ➡ multi partite information etc





higer genus

- Massive Thirring model

$$\mathcal{L}_F = i\,\bar{\psi}\,\gamma^{\mu}\partial_{\mu}\psi + \frac{\pi}{2}\lambda\,(\bar{\psi}\,\gamma^{\mu}\psi)(\bar{\psi}\,\gamma_{\mu}\psi) + m\,\bar{\psi}\psi$$

➡ mass perturbation (analytically tractable)

- Numerical approach [Marić, Bocini, Fagotti, 2023]

XXZ spin chain ➡ We can explore the large mass region.

Appendix

付録:ホログラフィー原理との関係

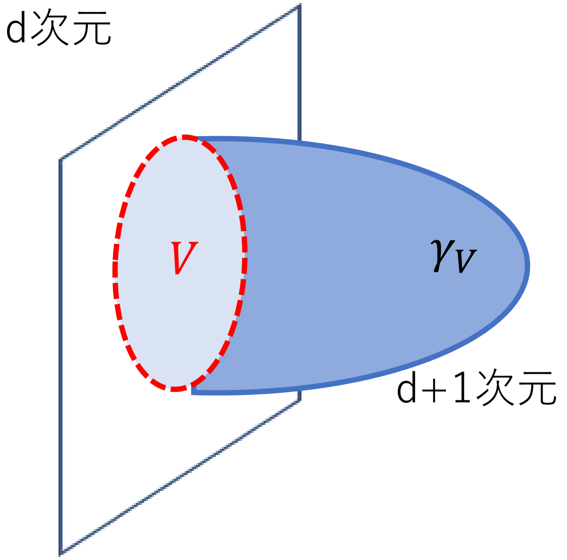
エンタングルメントエントロピーの面積則：

$$S(V) = \frac{1}{4\,G_N} A(\gamma_V)$$

[Ryu,Takayanagi 2006]

ブラックホールの面積則：

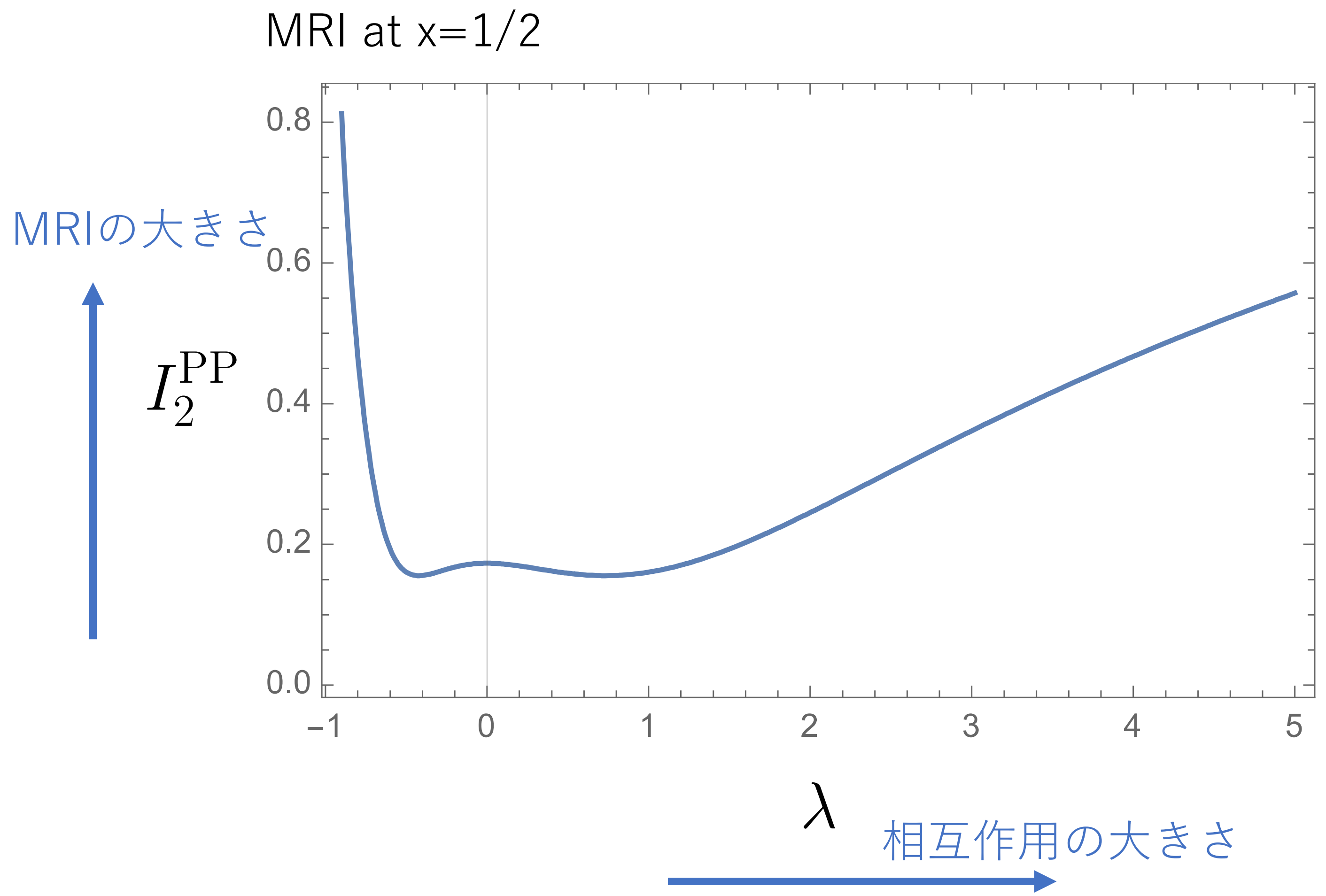
$$S_{BH} = \frac{k_B c^3}{4\hbar G_N} A$$

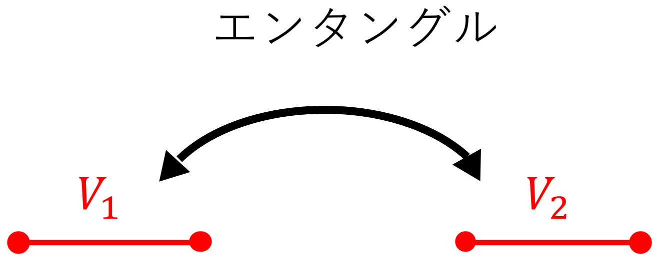


エンタングルメントはホログラフィー原理の研究に新たな切り口を与えた

付録: 相互Rényi情報量(MRI)の結合定数依存性

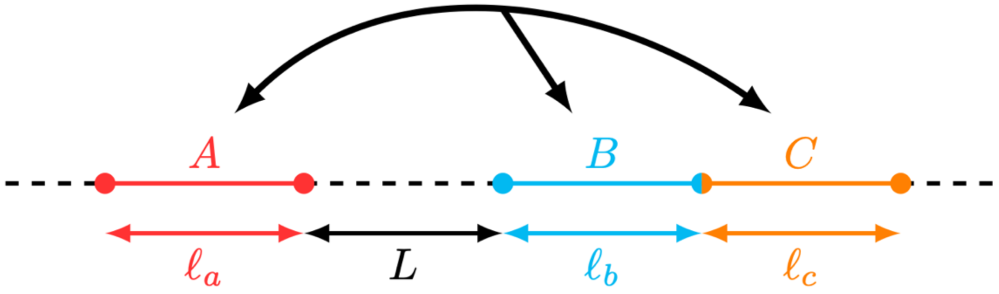
$$\text{MRI}：I_n(V_1, V_2) = S_n(V_1) + S_n(V_2) - S_n(V_1 \cup V_2)$$

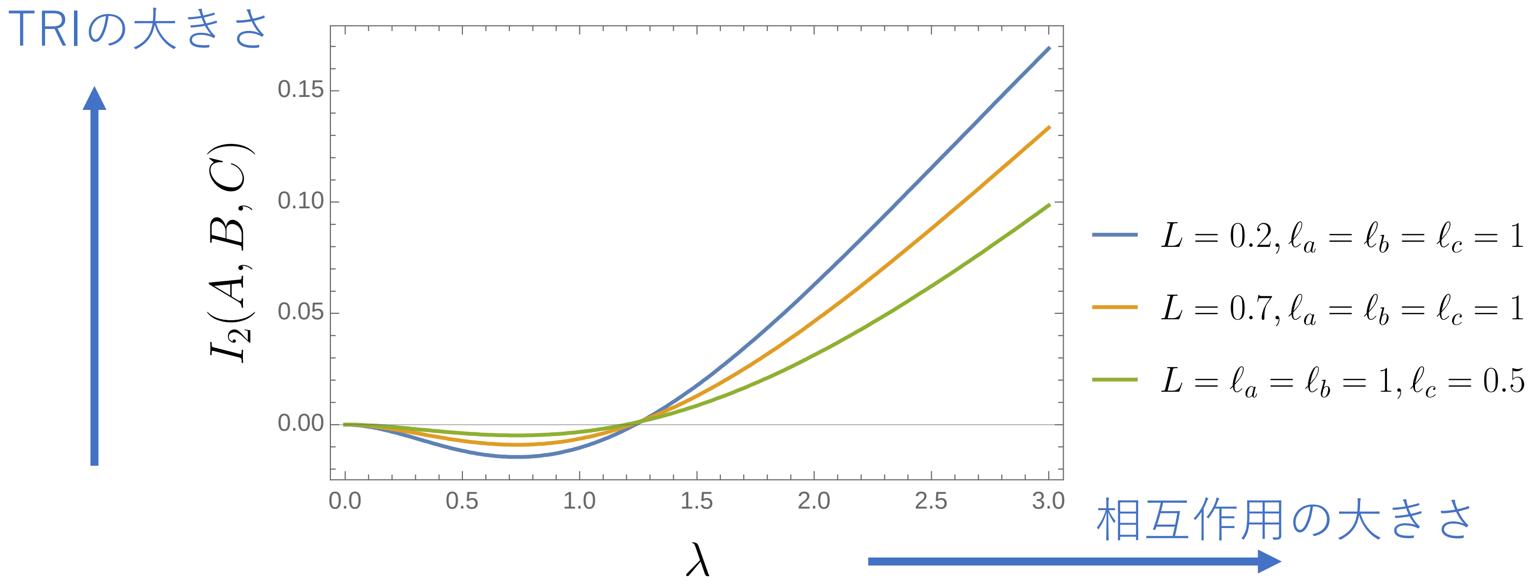




付録: トリパータイトRényi情報量(TRI)

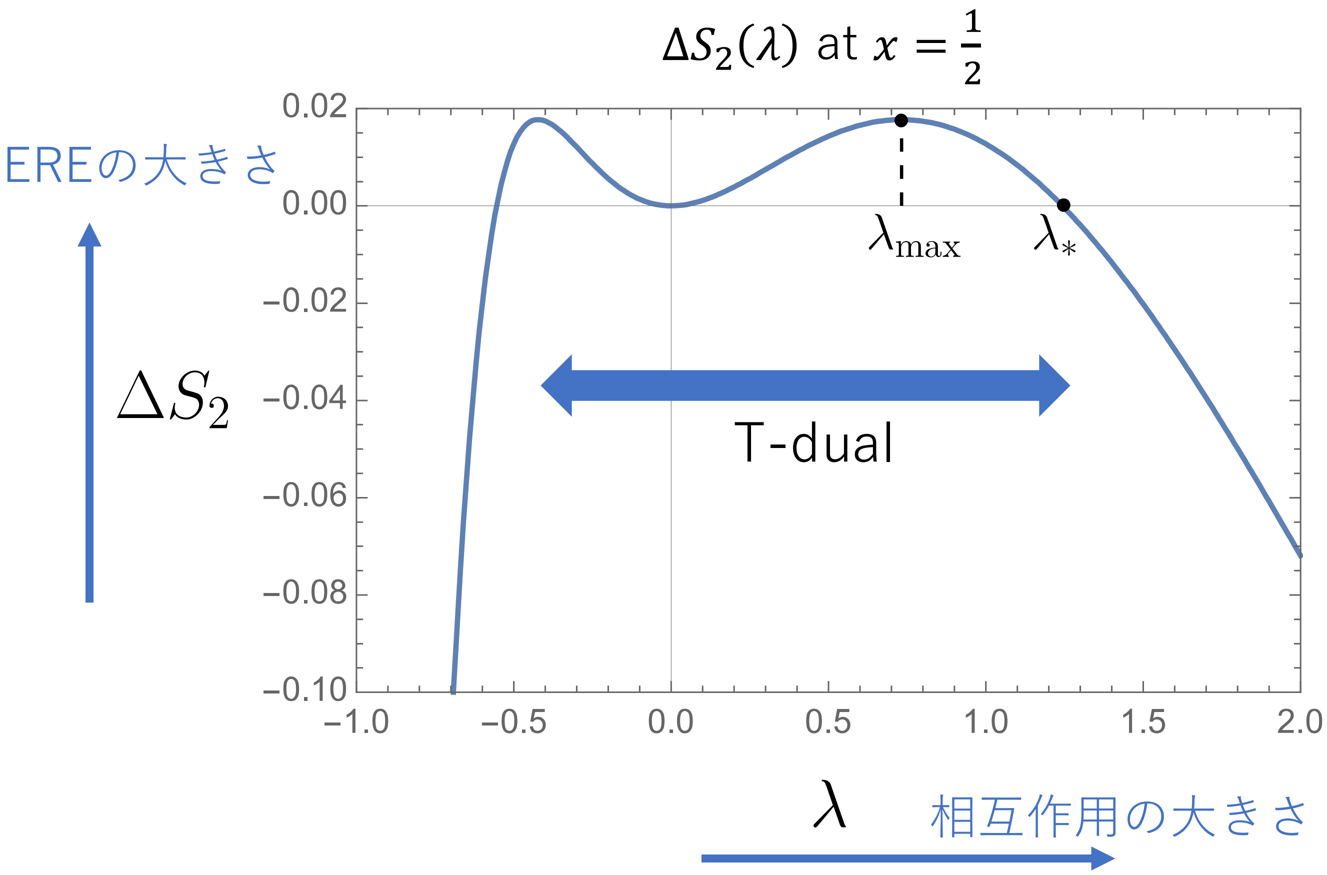
$$\text{TRI}: I_n(A,B,C) = S_n(A\cup B\cup C) - S_n(A\cup B) - S_n(B\cup C) - S_n(C\cup A) + S_n(A) + S_n(B) + S_n(C)$$





付録: T-dualityについて

$$\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$$



Compact bosonのT-duality：

$$R \to \frac{2}{R}$$

$$\lambda \to \lambda_{\mathrm{dual}} = -\frac{\lambda}{\lambda + 1}$$

$\lambda > 0$と$\lambda < 0$は互いに対応している

付録: T-dualityについて

Compact boson

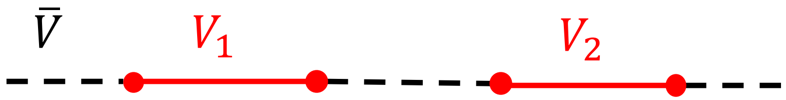
T-duality

$$
\begin{tikzcd}[column sep=large, row sep=huge]
S^1[R] \arrow[r, "\mathbb{Z}_2^{\mathcal{B}}", "\text{Orbifold}"'] \arrow[d, "\substack{\text{Fermionize} \\ \text{by } \mathbb{Z}_2^{\mathcal{B}}}"] & \dfrac{S^1[R]}{\mathbb{Z}_2^{\mathcal{B}}} = S^1\left[\dfrac{R}{2}\right] \arrow[d, "\substack{\text{Fermionize} \\ \text{by } \mathbb{Z}_2^{\mathcal{B}'}}"] \arrow[r, phantom, "="] & S^1\left[\dfrac{4}{R}\right] \arrow[d, "\substack{\text{Fermionize} \\ \text{by } \mathbb{Z}_2^{\mathcal{B}''}}"] \\
\mathcal{F}[\lambda] \arrow[r, leftrightarrow, "Arf[\rho]"] \arrow[rr, leftrightarrow, bend right, "Arf[\rho]"'] & \mathcal{F}'[\lambda] \arrow[r, phantom, "="] & \mathcal{F}[\tilde{\lambda}]
\end{tikzcd}
$$

Thirring model

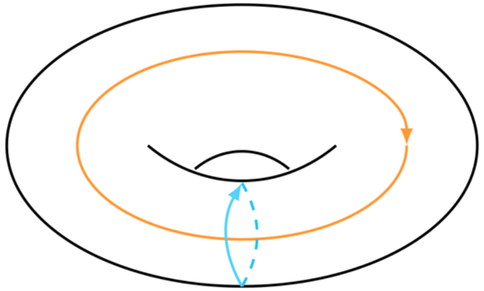
dual coupling : $\tilde{\lambda} = -\dfrac{\lambda}{1+\lambda}$

付録: cross-ratio x と トーラスのmoduli τ の関係



cross-ratio x

$$x=\left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$



moduli τ

簡単のため$\tau = i\,\ell$とおく。