# Entanglement Rényi entropy and boson-fermion duality in massless Thirring model

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# Outline

# Introduction

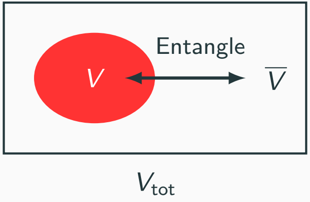
# 1. Entanglement in QFT

State of total system : $|0\rangle$

Reduced density matrix : $\rho_{\text{tot}} = |0\rangle\langle 0|$

Total space : $V_{\text{tot}} = V \cup \overline{V}$

Reduced density matrix : $\rho_V \equiv \text{Tr}_{\overline{V}}[\rho_{\text{tot}}]$




- Entanglement Rényi entropy (ERE) :

$$S_n(V) \equiv \frac{1}{1-n} \log \text{Tr}_V [\rho_V^n] \ , \ n \in \mathbb{Z}_+ \ ,$$

- Mutual Rényi information (MRI) :

$$I_n(V_1, V_2) \equiv S_n(V_1) + S_n(V_2) - S_n(V_1 \cup V_2) \ ,$$

$\Rightarrow$ Correlation between $V_1$ and $V_2$, where $V = V_1 \cup V_2$.

ERE and MRI are important quantities that characterize QFT.

# 1. Entanglement in QFT

However, Calculating $\mathrm{Tr}_V\left[\rho_V^n\right]$ in QFT is very difficult...

$$\mathrm{Tr}_V\left[\rho_V^n\right] \quad \sim \quad Z_n$$

Partition function

Known exact results (1+1 dimension) :


- ✓ Free massless fermion. $n$ sheets, $N$ intervals. [Casini, Fosco, Huerta 2005]
- ✓ Arbitrary CFT. But single interval only. [Holzhey,Larsen, Wilczek 1994]
- ✓ Free compact boson. 2 sheets, 2 intervals. [Calabrase, Cardy, Tonni 2011]

There is no exact result for interacting system.

# 1. Boson-fermion duality

It's difficult to compute the ERE and MRI in interacting system.

$\Rightarrow$ We focused on boson-fermion duality. [Karch, Tong, Turner 2019]

| **fermionic theory $\mathcal{T}_{\mathsf{F}}$** | | **bosonic theory $\mathcal{T}_{\mathsf{B}}$** |
|---|---|---|
| discrete sym : $\mathbb{Z}_2^{\mathsf{F}}$ | fermionization $\longleftarrow$ | discrete sym : $\mathbb{Z}_2^{\mathsf{B}}$ |
| local op : $\hat{\mathcal{O}}^F$ | $\longrightarrow$ bosonization | local op : $\hat{\mathcal{O}}^B$ |
| partition function : $Z^F$ | | partition function : $Z^B$ |

fermionization dictionary (1+1 dim, closed manifold) :

couple with the TQFT

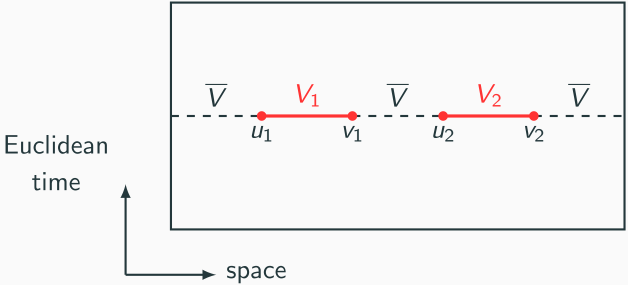
$$\mathcal{T}_{\mathsf{F}} = \frac{\mathcal{T}_{\mathsf{B}} \times \mathrm{Kitaev}}{\mathbb{Z}_2^{\mathsf{B}}}$$

gauging $\mathbb{Z}_2^{\mathsf{B}}$ sym

# 1. Our work

**What we did**

- Massless Thirring model (1+1 dim, fermion, 4-point interaction)
- Combined conventional method (Replica method) and boson-fermion duality
- Got exact results on ERE and MRI in the case of the figure bellow
- Saw parameter dependence of ERE and MRI




$$S_2(V)$$

$$I_2(V_1, V_2)$$

# Analysis of the ERE and MRI

# The model

**Massless Thirring model**

$$\mathcal{L}_{\mathsf{F}} = \mathrm{i}\,\bar{\psi}\,\partial\!\!\!/\,\psi + \underbrace{\frac{\pi}{2}\,\lambda\,(\bar{\psi}\,\gamma^{\mu}\,\psi)(\bar{\psi}\,\gamma_{\mu}\,\psi)}_{\text{interaction}}$$

$\psi$ : Dirac fermion

$\mathbb{Z}_2^{\mathsf{F}}$ : $\psi \to -\psi$

$\lambda$ : Thirring coupling

difficult to analyze

fermionize ←

$$1 + \lambda = \frac{4}{R^2}$$

**Free compact boson**

$$\mathcal{L}_{\mathsf{B}} = -\frac{R^2}{8\pi}\partial_{\mu}\phi\partial^{\mu}\phi$$

$$\phi \sim \phi + 2\pi$$

$\phi$ : scalar

$\mathbb{Z}_2^{\mathsf{B}}$ : $\phi \to \phi + \pi$

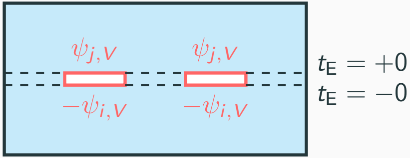
$R$ : compact boson radius

easy to analyze

We fermionize compact boson and get massless Thirring model.

# Repica method

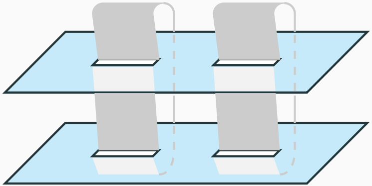
Target : $S_2(V) = -\log \mathrm{Tr}_V\,[\rho_V^2]$




$$\rho_V(\psi_i, \psi_j) = \mathrm{Tr}_{\overline{V}}\left[\langle \psi_{j,V}|0\rangle \langle 0|\psi_{i,V}\rangle\right]$$

$$\mathrm{Tr}_V\left[\rho_V^2\right] = \sum_{\psi_1,\psi_2} \rho_V(\psi_1,\psi_2)\,\rho_V(\psi_2,\psi_1)$$

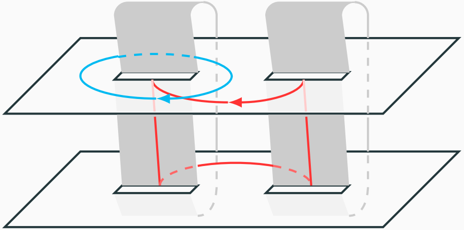
$$\sim Z^F_{\Sigma_{2,2}}$$



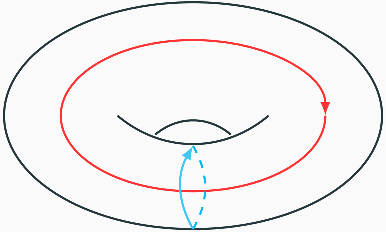
$\Sigma_{2,2}$

➡ We have to calculate the partition function $Z^F_{\Sigma_{2,2}}$

# Conformal map [Lumin, Mathur 2001]



$\Longleftrightarrow$



$\Sigma_{2,2}$ $\qquad$ $\mathbf{T}$

cross-ratio : $x = \frac{(v_1 - u_1)(v_2 - u_2)}{(u_2 - u_1)(v_2 - v_1)}$ $\longleftrightarrow$ moduli of torus : $\tau$

$$Z^F_{\Sigma_{2,2}}(x) = (\text{UV div})\ f(x)\ Z^F_{\mathbf{T}}(\tau)$$

We can map to torus

# Analytical result

$$S_2(V) = -\log f(x)(\text{UV div}) \ -\log \ \boxed{Z^F_{\mathbf{T}}}$$

fermionization

$$S_2(V) = -\log(\text{UV div}) \ -\log \ \boxed{\left(\text{sum of } Z^B_{\mathbf{T}}[t]\text{s}\right)} \longleftarrow \text{free theory (known)}, \ t : \mathbb{Z}_2^B \text{ gauge field}$$

**Exact result**

$$S_2(V,\lambda) = S_2(V,0) - \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2\left(\tau(1+\lambda)\right)\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]$$

where,

$S_2(V,0)$ : Free ERE $\rightarrow$ consistent with existing result

$\vartheta_j(\tau)$, $j = 2,3,4$ : Jacobi theta functions

We got the exact ERE and MRI in massless Thirring model

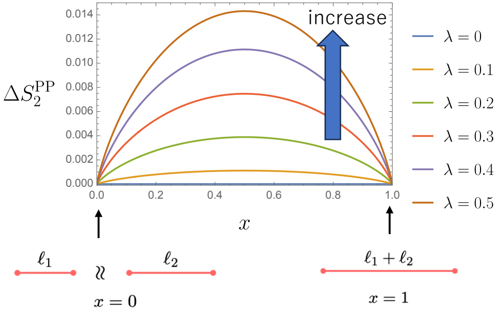
# Parameter dependence of the ERE and MRI

# Cross-ratio dependence of ERE

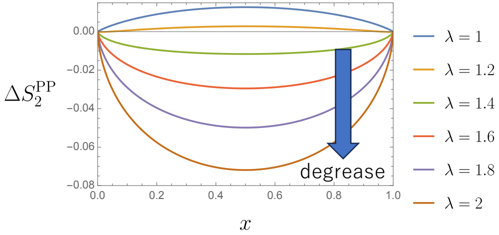
Deviation : $\Delta S_2(x) = S_2(V,\lambda) - S_2(V,0)$

ERE

Small $\lambda$




Large $\lambda$



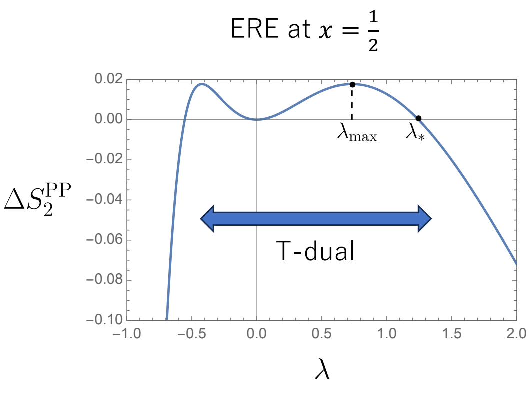

Single interval, CFT [Holzhey et al 1994]

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left(\frac{v-u}{\epsilon}\right)$$

$c$ : central charge,
$\epsilon$ : UV cutoff

➡ We got reasonable results

# Coupling dependence of ERE



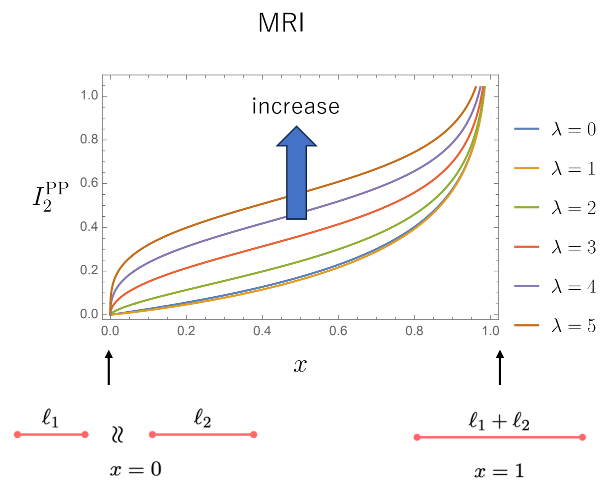

T-duality :

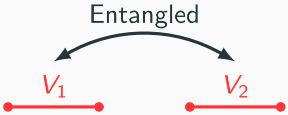
$$\lambda \to \lambda_{\mathsf{dual}} \equiv -\frac{\lambda}{\lambda+1}$$

$\lambda > 0$ and $\lambda < 0$ correspond

We investigated the behavior with respect to the coupling constants.

# Cross-ratio dependence of MRI




As coupling $\lambda$ becomes larger, the entanglement between $V_1$ and $V_2$ grows stronger.

# Summary

### Summary

- ERE and MRI are important quantities that characterize QFT.
- We exactly derived the ERE and MRI in an interacting system by fermionization.
- We saw the behaviors of ERE and MRI from exact results.

### Future work

- ERE and MRI for 3-intervals or more.
- Massive Thirring model.
- Other quantum information measures.

**Thank you.**

# Backup slides : The duality web

Compact boson

$$\begin{tikzcd}[column sep=large]
S^1[R] \arrow[r, "\mathbb{Z}_2^{\mathcal{B}}", "\text{Orbifold}"'] \arrow[d, "\text{Fermionize by } \mathbb{Z}_2^{\mathcal{B}}"] & \dfrac{S^1[R]}{\mathbb{Z}_2^{\mathcal{B}}} = S^1\left[\dfrac{R}{2}\right] \arrow[r, leftrightarrow, "\text{T-duality}"] \arrow[d, "\text{Fermionize by } \mathbb{Z}_2^{\mathcal{B}'}"] & S^1\left[\dfrac{4}{R}\right] \arrow[d, "\text{Fermionize by } \mathbb{Z}_2^{\mathcal{B}''}"'] \\
\mathcal{F}[\lambda] \arrow[r, leftrightarrow, "Arf{[\rho]}"] \arrow[rr, leftrightarrow, bend right, "Arf{[\rho]}"'] & \mathcal{F}'[\lambda] \arrow[r, leftrightarrow, "\text{equivalent}"] & \mathcal{F}[\tilde{\lambda}]
\end{tikzcd}$$

Thirring model

dual coupling : $\tilde{\lambda} = -\dfrac{\lambda}{1+\lambda}$

# Backup slides : Coupling dependence of ERE