A Scalable Quantum Algorithm for Quantum Mpemba Effect

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Based on ongoing work with
Keisuke Fujii, Masazumi Honda and Duc Truyen Le.

1. Introduction

The main topic of this talk:

Quantum computing → apply → **Symmetry restoration dynamics**

Take home message :

Quantum computing enables the study of out-of-equilibrium symmetry restoration dynamics, such as Quantum Mpemba Effect.

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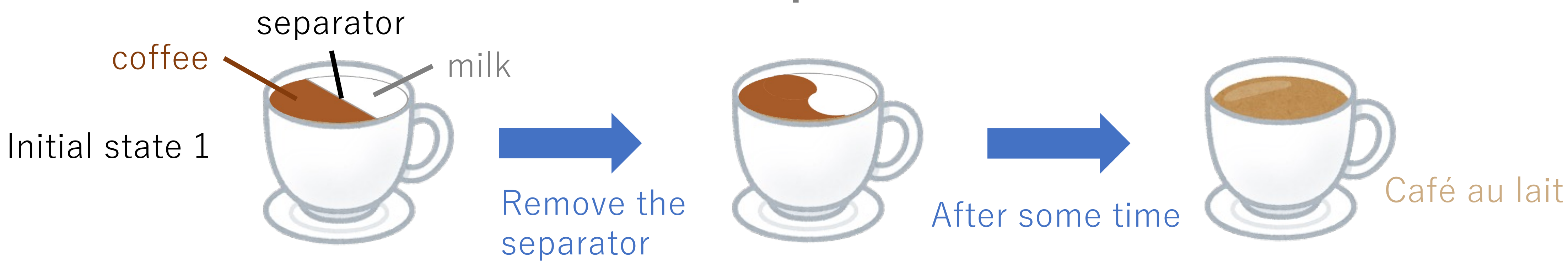
Quantum computing enables the study of out-of-equilibrium symmetry restoration dynamics, such as Quantum Mpemba Effect.

↑ What is this ?

1. Introduction

Mpemba effect is an anomalous out-of-equilibrium relaxation.
[E B Mpemba and D G Osborn, 1969]

Metaphor



1. Introduction

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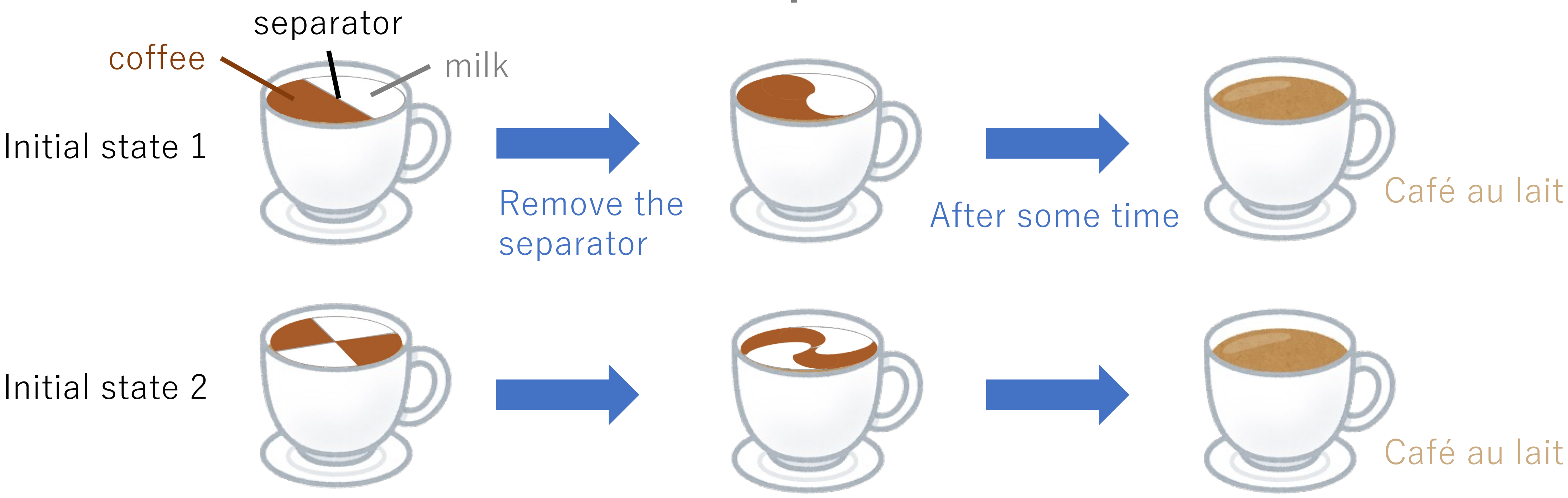


Intuitively, one expects the initial state 2 to relax faster than the initial state 1.

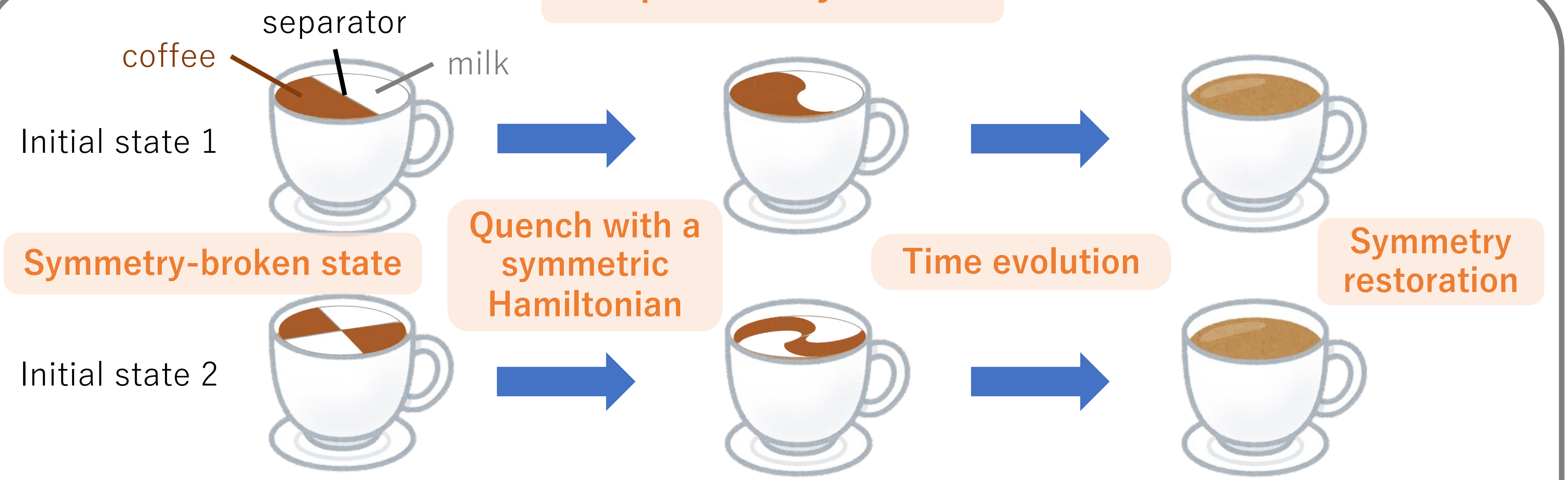
However, under certain conditions, the opposite behavior is observed.

➡ Counterintuitive phenomenon!

1. Introduction

Mpemba effect is an anomalous out-of-equilibrium relaxation.
[E B Mpemba and D G Osborn, 1969]

In quantum system…



Quantum Mpemba effect

The more symmetry is initially broken, the faster it is restored.

1. Introduction

How to quantify the degree of symmetry breaking on subsystem?

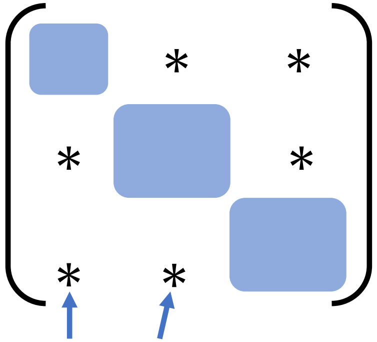
We assume that the theory has a symmetry with charge

$$Q = Q_A + Q_B$$

A = Subsystem of interest, B = Environment.

Symmetry-broken state

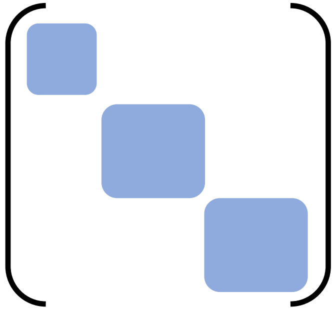
$\rho_A =$



Off-diagonal elements in eigenbasis of Q_A

Symmetric state

$\rho_{A,S} =$



Block diagonal

Entanglement Asymmetry : A quantifier of symmetry breaking

[Ares-Murciano-Calabrese, 2022]

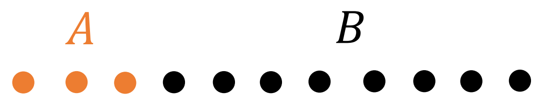
$$\Delta S_A \equiv \Delta S(\rho_A | \rho_{A,S}) = \mathrm{Tr}_A[\rho_A(\log \rho_A - \log \rho_{A,S})]$$

Relative entropy

1. Introduction

Typical protocol to investigate the quantum Mpemba effect.

Quench dynamics with a symmetric Hamiltonian



Spin chain or QFT

Total system= $A \cup B$

A : subsystem of interest

STEP1: Prepare an initial state $|\psi_{AB}(0)\rangle$ that **explicitly** breaks the symmetry.

STEP2: Perform the time evolution.

$$|\psi_{AB}(t)\rangle = e^{-iHt}|\psi_{AB}(0)\rangle,$$ where H is a symmetric Hamiltonian.

STEP3: Compute the entanglement asymmetry at time t.

$$\Delta S_A(t)$$

The entanglement asymmetry captures the real-time dynamics of symmetry restoration.

1. Introduction

The quantum Mpemba effect has been extensively studied in various fields, including condensed matter physics, high-energy physics, and quantum information.

However, several challenges remain···

Challenges

- Computing the entanglement asymmetry beyond CFTs or integrable systems.
- Numerical approaches by classical computer : $\mathcal{O}(2^{N_B} + 2^{3N_A})$ → **exponentially costly**

  $N_{A/B}$: the size of subsystem A/B
- Monte Carlo methods suffer from the sign problem in real-time dynamics.

Large systems, including quantum field theories, are therefore difficult to access...

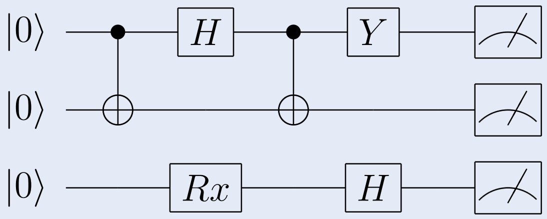
↓

We need a new scalable method to investigate the quantum Mpemba effect.

1. Introduction

Our approach:

Quantum computing



Simulation →

Quantum Mpemba Effect

$\Delta S_A(t)$

Entanglement Asymmetry

Short summary of our work

- We propose a **scalable** quantum algorithm that efficiently compute the entanglement asymmetry.
- As an application, we study Schwinger model and demonstrate that our quantum algorithm can be used to investigate the quantum Mpemba effect in quantum field theories.
- Finally, we estimate the resources required to implement the quantum algorithm and show its scalability.

Outline

1. Introduction
2. Quantum Algorithm for Entanglement Asymmetry
3. Quantum Mpemba Effect in Schwinger Model
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2. Quantum Algorithm for Entanglement Asymmetry

To simplify the analysis, we focus on the Rényi entanglement asymmetry:

$$\Delta S_A^{(n)} \equiv \frac{1}{n-1}\left(\log \mathrm{Tr}_A[\rho_A^n] - \log \mathrm{Tr}_A\left[\rho_{A,S}^n\right]\right), \quad \lim_{n\to 1} \Delta S_A^{(n)} = \Delta S_A$$

For concreteness, we consider $n = 2$.

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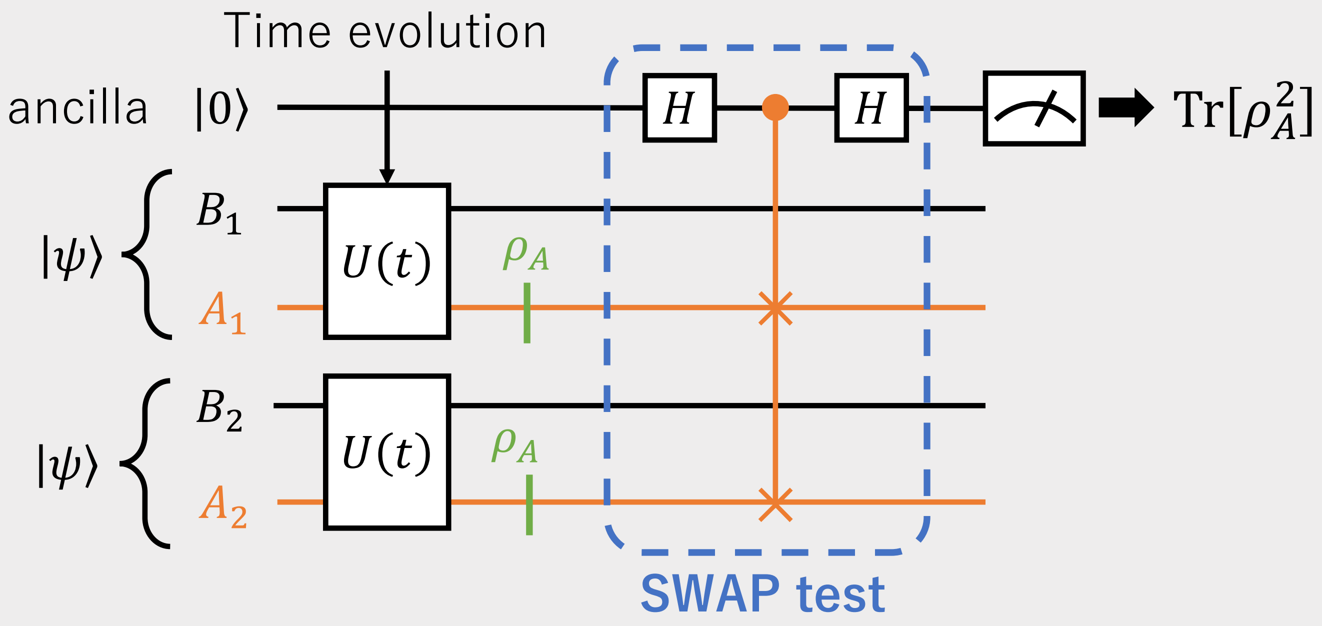
Quantum circuit for computing $\mathrm{Tr}_A[\rho_A^2]$

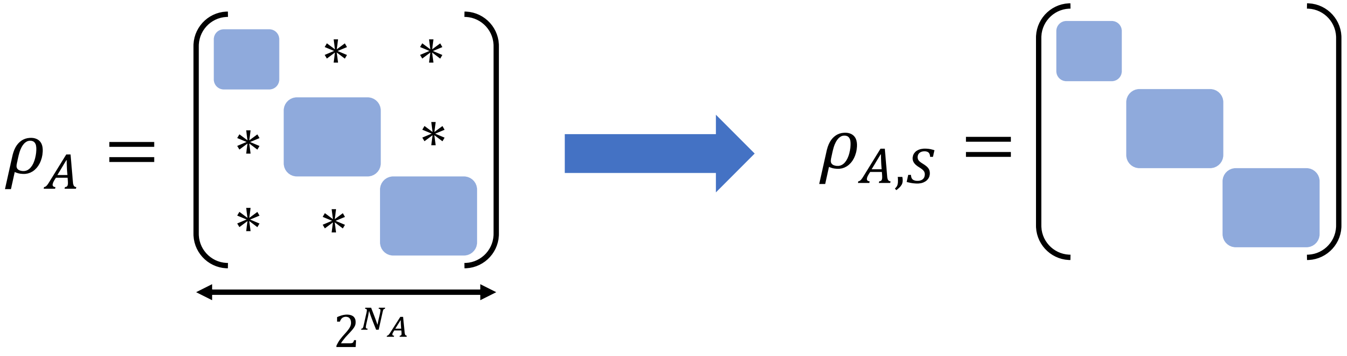


How can we realize $\rho_{A,S}$ in a quantum circuit and compute $\mathrm{Tr}_A[\rho_{A,S}^2]$?

(Our work)

2. Quantum Algorithm for Entanglement Asymmetry

Naively, one need to remove $\mathcal{O}(2^{2N_A})$ off-diagonal elements.



, in eigenbasis of Q_A

How can this be done efficiently in quantum circuit?

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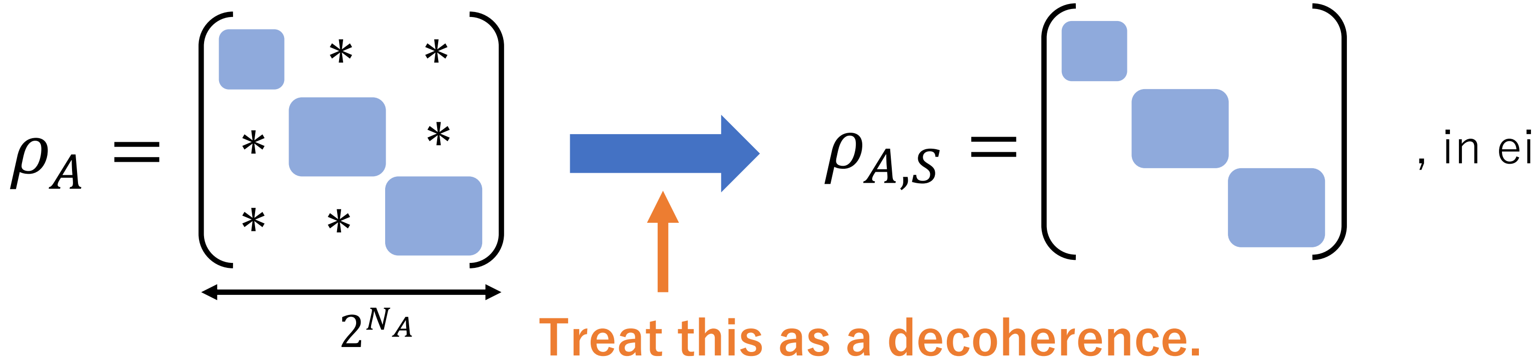
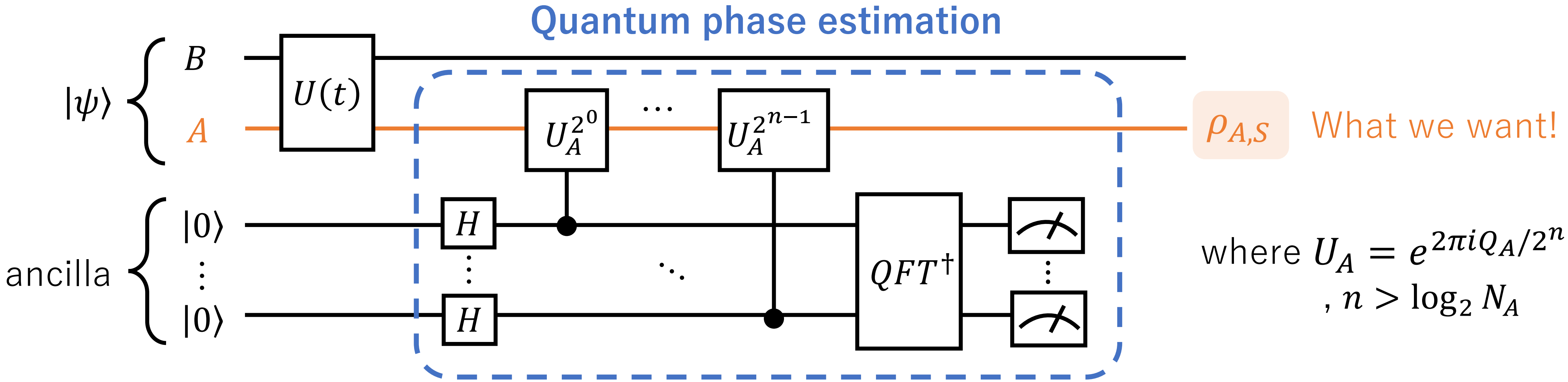


How can this be done efficiently in quantum circuit?



where $U_A = e^{2\pi i Q_A/2^n}$, $n > \log_2 N_A$

$\rho_{A,S}$ can be realized by employing quantum phase estimation.

2. Quantum Algorithm for Entanglement Asymmetry

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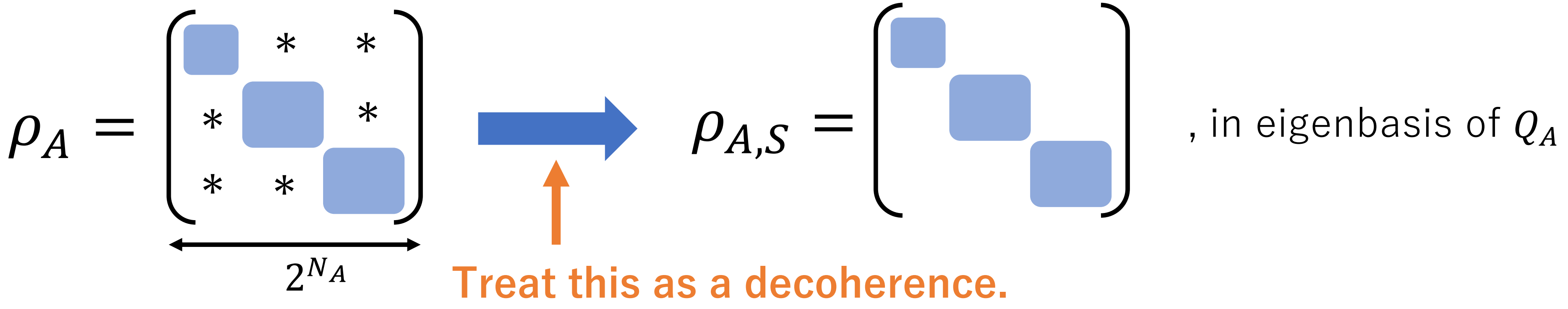
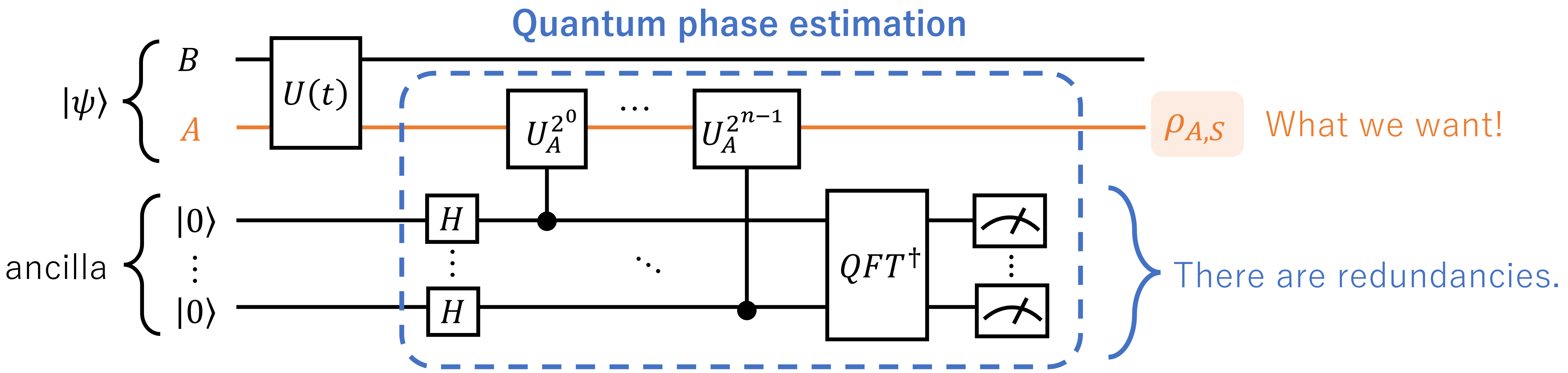


How can this be done efficiently in quantum circuit?



2. Quantum Algorithm for Entanglement Asymmetry

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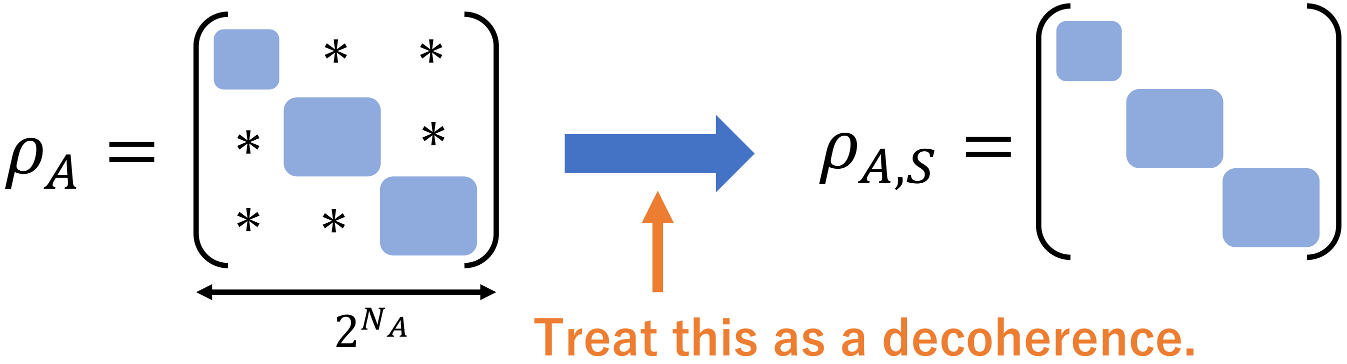
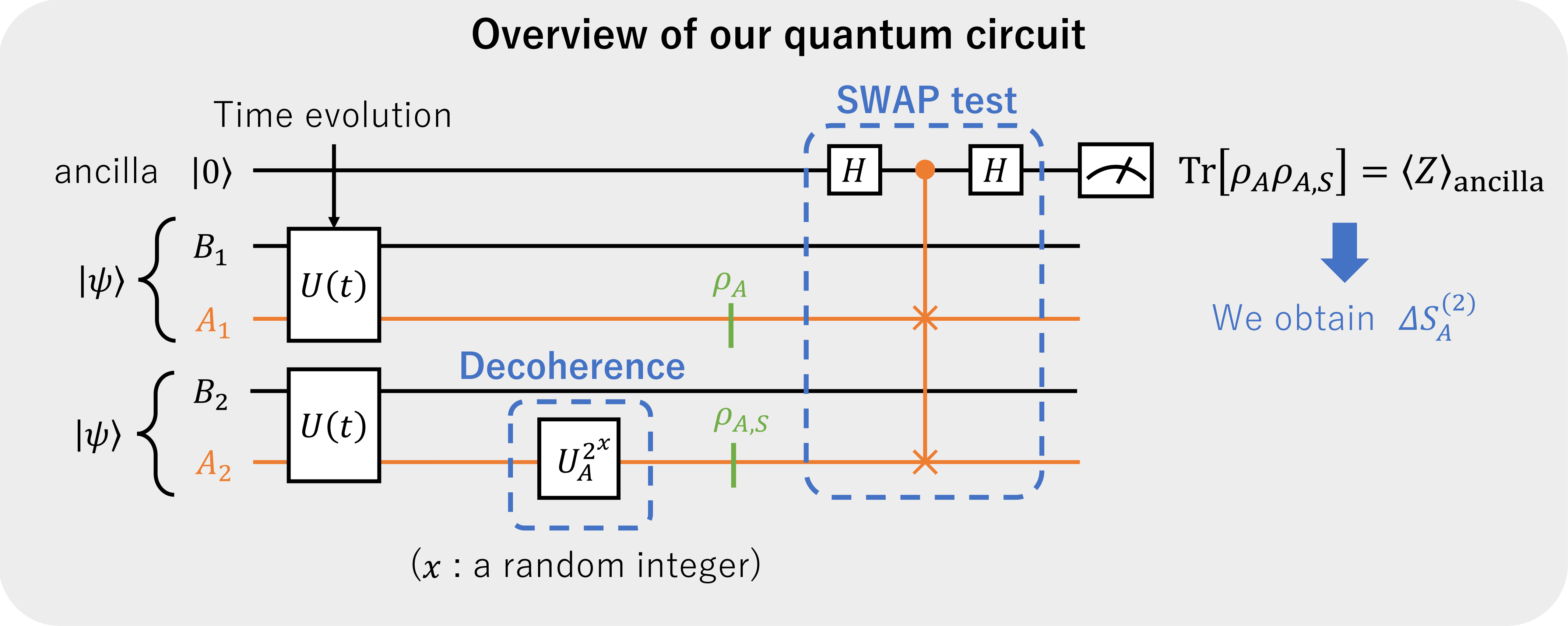


, in eigenbasis of Q_A

How can this be done efficiently in quantum circuit?

We can drastically simplify the quantum circuit.

2. Quantum Algorithm for Entanglement Asymmetry



By combining the idea of quantum phase estimation with SWAP test, we can estimate the Rényi Entanglement Asymmetry.

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3. Quantum Mpemba Effect in Schwinger Model

Schwinger model in the continuum [Schwinger, 1962]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_\mu + \mathrm{i}gA_\mu)\psi - m\bar{\psi}e^{\mathrm{i}\theta\gamma^5}\psi$$

Temporal gauge $A_0 = 0$

Kogut-Susskind formalism

Open boundary condition

Gauss law constraint

Jordan-Wigner transformation

Schwinger model as a spin model [Masazumi Honda et al, 2022]

$$H = H_{ZZ} + H_{\pm} + H_Z$$

$$H_{ZZ} \sim (\mathrm{const}) \sum_n \sum_{1\le k<\ell\le n} Z_k Z_\ell \ , \quad H_{\pm} \sim \sum_n (\mathrm{const})(X_n X_{n+1} + Y_n Y_{n+1}) \ , \quad H_Z = \sum_n (\mathrm{const})\, Z_n$$

X_n, Y_n, Z_n : Pauli matrices on the n-th site.

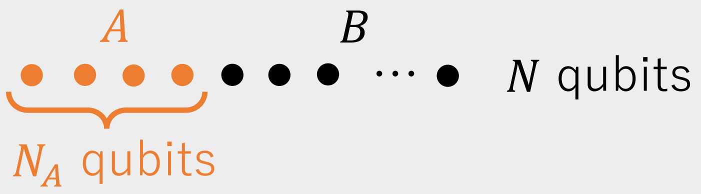
3. Quantum Mpemba Effect in Schwinger Model

Our setup:

Symmetry

$U(1)$ symmetry, $Q = \frac{1}{2}\sum_n Z_n$

Configuration of subsystem



The procedure of our simulation:

STEP1: Prepare the following initial state.

$$|\phi\rangle = e^{-\mathrm{i}\frac{\phi}{2}\sum_n Y_n Y_{n+1}} |\uparrow\uparrow\cdots\uparrow\rangle \quad , \phi : \text{Parameter of initial symmetry breaking.}$$

STEP2: Perform the time evolution by second order Trotterization

$$H = H_{ZZ} + H_{\pm} + H_Z, \qquad H_{\pm} = H_{\pm,\mathbf{even}} + H_{\pm,\mathbf{odd}}$$

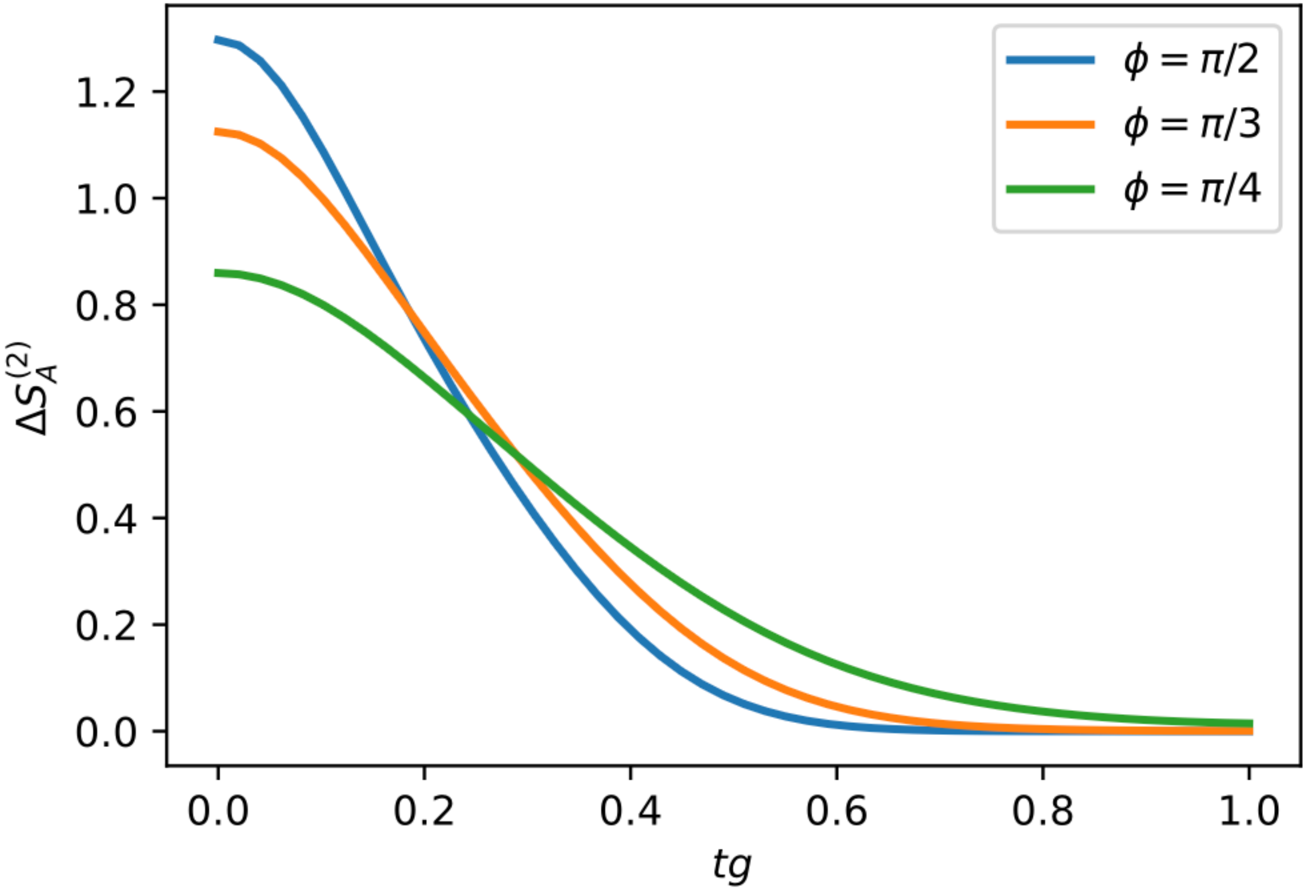
STEP3: Compute $\Delta S_A^{(2)}(t)$ which is obtained by our algorithm.

We performed above quantum simulations using Qulacs, a Python library for classical simulation.

3. Quantum Mpemba Effect in Schwinger Model

An example of our result:

Degree of symmetry breaking



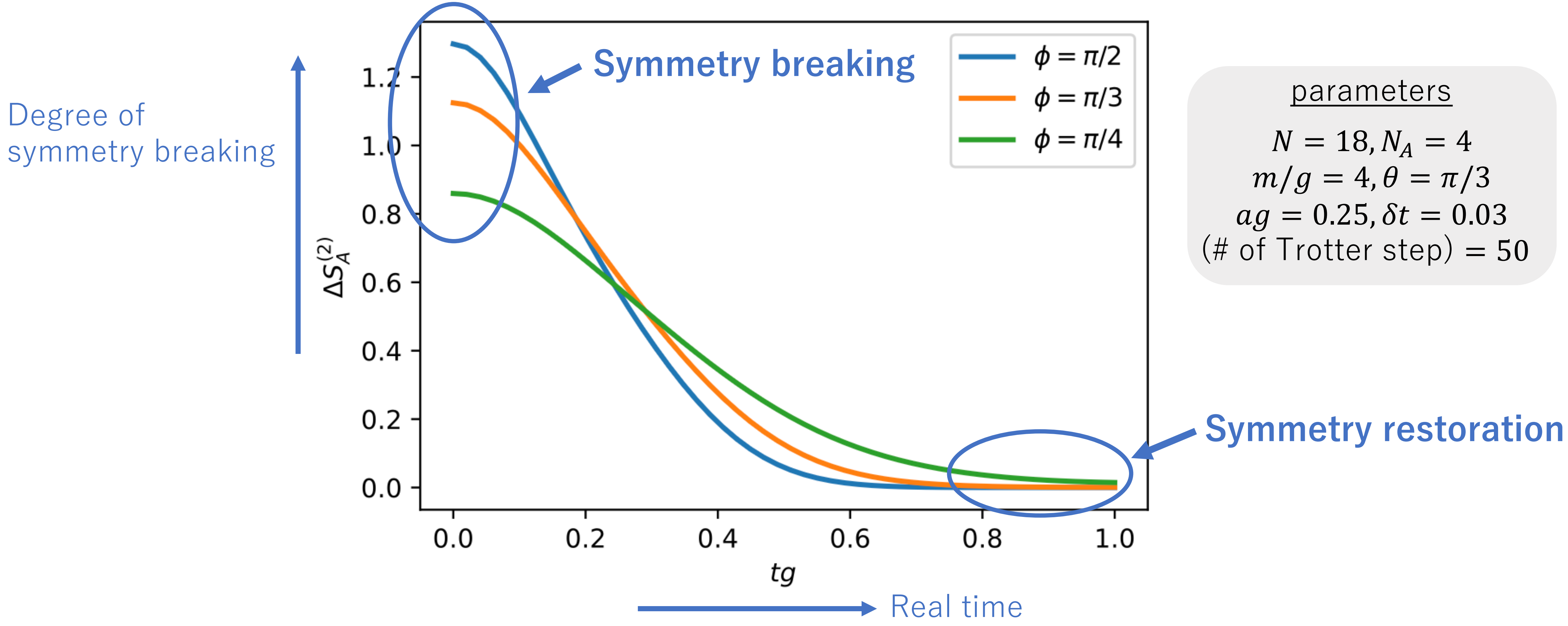
Real time

parameters

$N = 18, N_A = 4$
$m/g = 4, \theta = \pi/3$
$ag = 0.25, \delta t = 0.03$
(# of Trotter step) $= 50$

3. Quantum Mpemba Effect in Schwinger Model

An example of our result:



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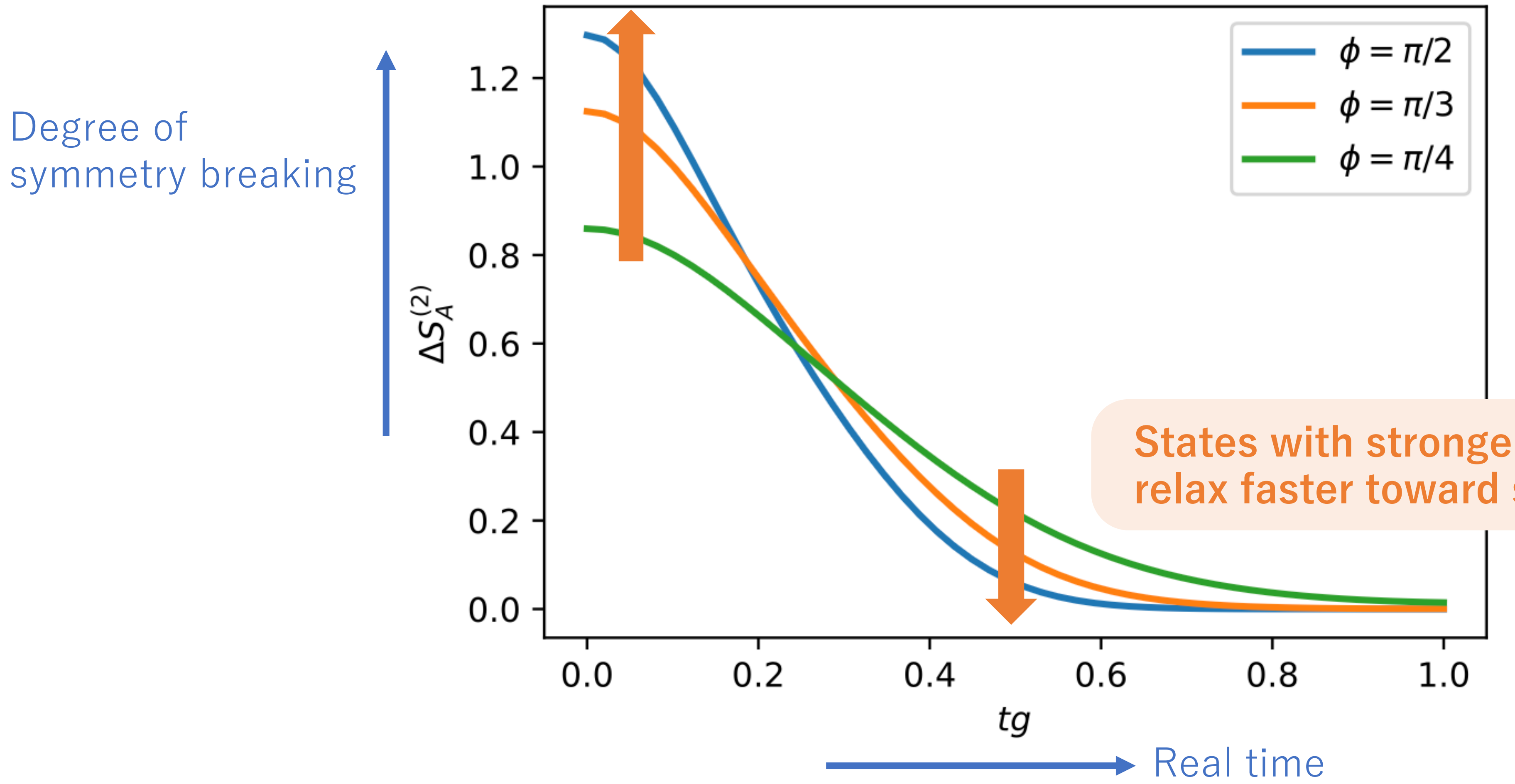

parameters

$N = 18, N_A = 4$
$m/g = 4, \theta = \pi/3$
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$(\text{\# of Trotter step}) = 50$

States with stronger symmetry breaking relax faster toward symmetry restoration.

Using our quantum algorithm, we demonstrate the quantum Mpemba effect in the Schwinger model.

3. Quantum Mpemba Effect in Schwinger Model

Resource estimation of our algorithm

There are two errors:

Statistical error of $\langle Z \rangle_{\text{ancilla}}$ in SWAP test

Required number of measurements N_{shot} :

$N_{\text{shot}} = \mathcal{O}(\delta^{-2})$, where δ is standard deviation

Does not depend on system size!

Systematic error of Trotterization

There is an error $\epsilon = \left\| e^{iHt} - U^M(t/M) \right\|$, where $U(t/M)$ is one step of Trotterization.

Required number of gates to achieve an error tolerance ϵ :

(gate complexity) $\leq \mathcal{O}\left(N^{9/2} t^{3/2} \epsilon^{-1/2}\right)$ Theoretical upper bound [A. M. Childs et al, 2021]

No exponential cost!

Our quantum algorithm is scalable and applicable to large systems.

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Summary

- The quantum Mpemba effect is an anomalous out-of-equilibrium phenomenon that has been extensively studied recently, but analyzing this phenomenon remains challenging.
- In this work, **we propose an efficient quantum algorithm for computing the entanglement asymmetry**.
- We demonstrate the quantum Mpemba effect in the Schwinger model.
- **Our algorithm is scalable and applicable to large systems**.

Future directions

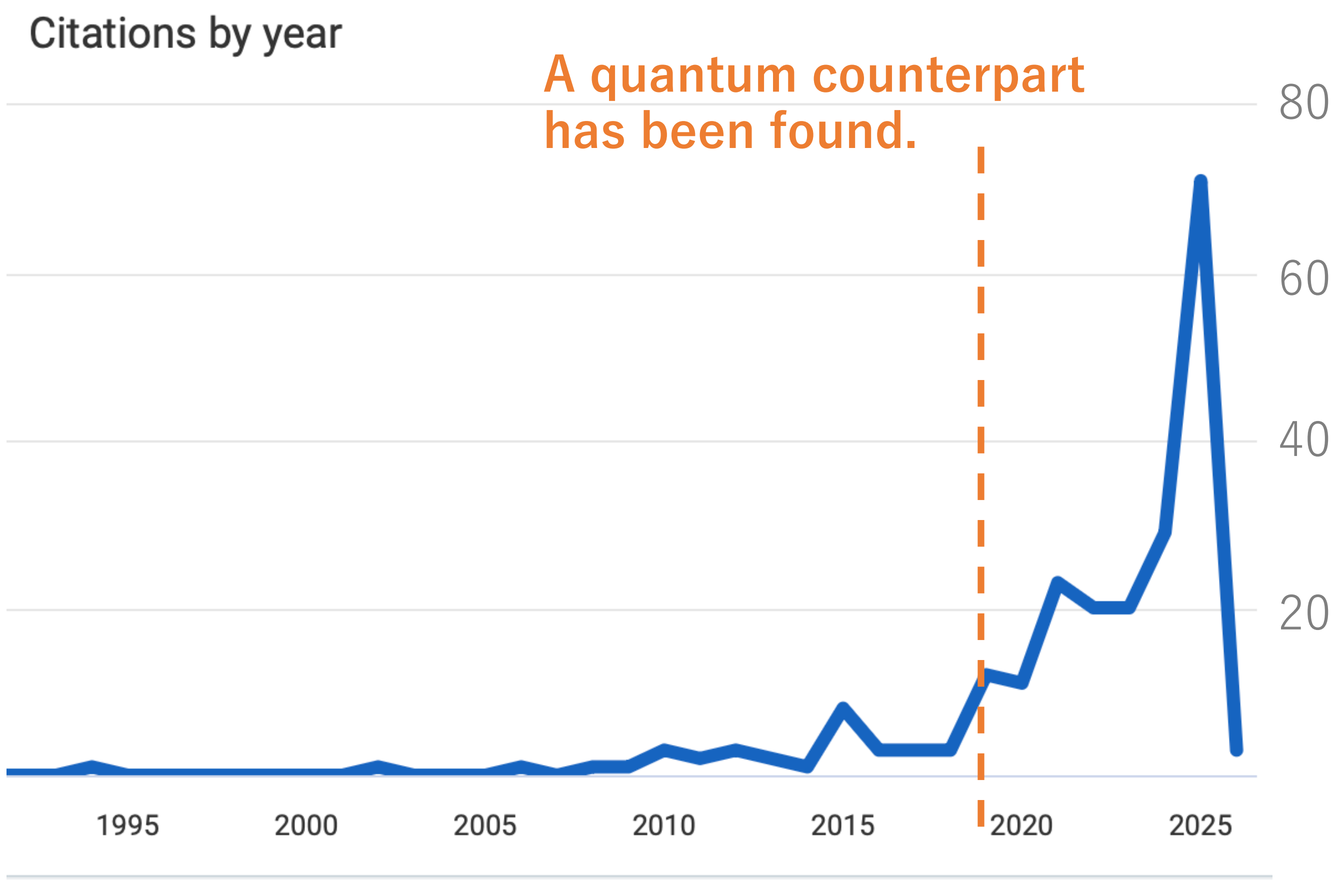
- The theoretical upper bound on gate complexity is overestimating.

  →We are currently evaluating tighter gate complexity bounds through numerical simulations (ongoing)
- Continuum limit
- Application to other models (our algorithm works for general quantum state)

Appendix

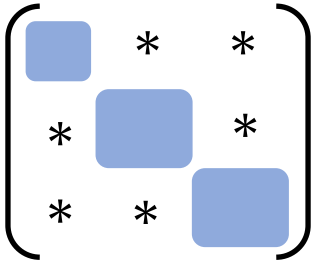
Citations by year

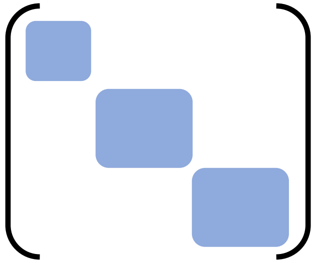
The Mpemba effect was first observed in classical systems, but a quantum counterpart has been found.
[E B Mpemba and D G Osborn, 1969]

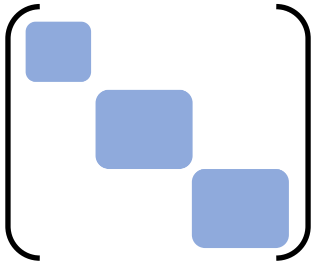
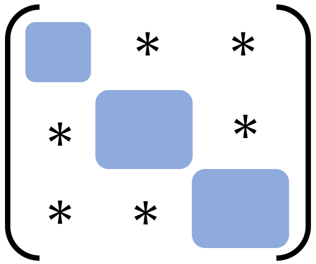


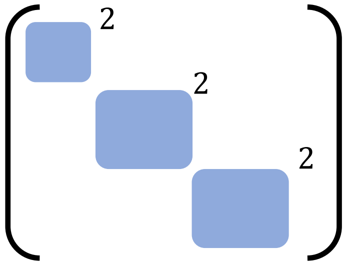
The quantum Mpemba effect is a rapidly developing topic!

Technical detail

$\rho_A =$ 

$\rho_{A,S} =$  , in eigenbasis of Q_A

$\mathrm{Tr}[\rho_A \rho_{A,S}] =$ Tr  

$=$ Tr 

$= \mathrm{Tr}[\rho_{A,S}^2]$

In general, the following identity holds

$$\mathrm{Tr}[\rho_A \rho_{A,S}^{n-1}] = \mathrm{Tr}[\rho_{A,S}^n]$$

Detail of Hamiltonian of Schwinger model

Schwinger model as a spin model

$$H = H_{ZZ} + H_{\pm} + H_Z$$

$$H_{ZZ} = \frac{J}{2}\sum_{n=2}^{N-1}\sum_{1\le k<\ell\le n} Z_k Z_\ell \ , \qquad H_{\pm} = \frac{1}{2}\sum_{n=1}^{N-1}\left(w - (-1)^n \frac{m}{2}\sin\theta\right)(X_n X_{n+1} + Y_n Y_{n+1})$$

$$H_Z = \frac{m\cos\theta}{2}\sum_{n=1}^{N}(-1)^n Z_n - \frac{J}{2}\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell}^{n} Z_\ell \quad , \quad J = \frac{g^2 a}{2}, w = \frac{1}{2a}$$

, a : lattice spacing

Initial state and EA

Symmetry

$U(1)$ symmetry, $Q = \frac{1}{2}\sum_n Z_n$

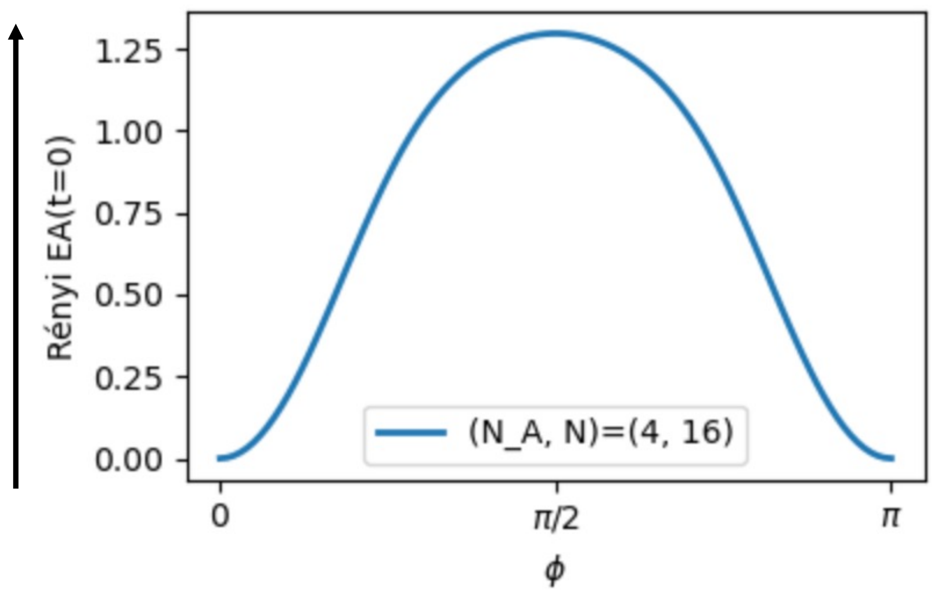
: Rotational symmetry around Z axis

Initial state

$$|\phi\rangle = e^{-i\frac{\phi}{2}\sum_n Y_n Y_{n+1}}|\uparrow\uparrow\cdots\uparrow\rangle$$

ϕ : Parameter of initial symmetry breaking.

Degree of symmetry breaking at $t = 0$



Initial angle ϕ

$\Delta S_A^{(2)}(t)$ **takes maximal value at** $\phi = \pi/2$ **and increasing function for** $\phi \in \left[0, \frac{\pi}{2}\right]$

Two choices of Trotterization

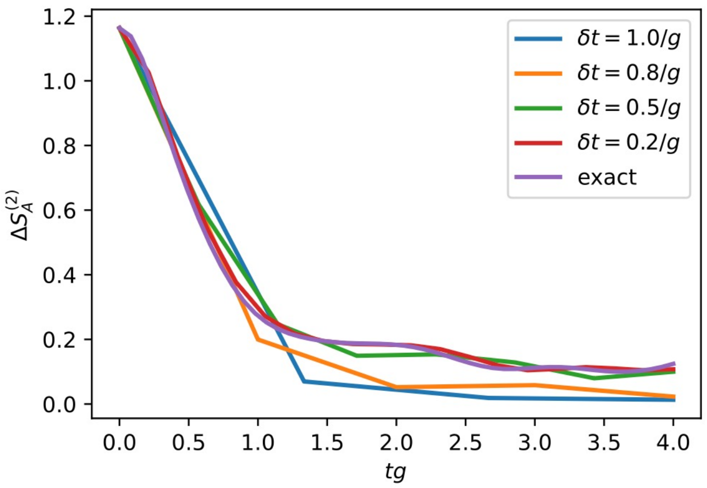
Second-order product formula:

$$H = \sum_{k=1}^{K} h_k\,,\ \ U(\delta t) = \prod_{k=1}^{K} e^{-i\frac{\delta t}{2}h_k} \prod_{k=K}^{1} e^{-i\frac{\delta t}{2}h_k} + \mathcal{O}(\delta t^3),\ \text{where } \prod_{k=i}^{j} V_k \equiv V_i \cdots V_j$$

$U(1)$ symmetric decomposition

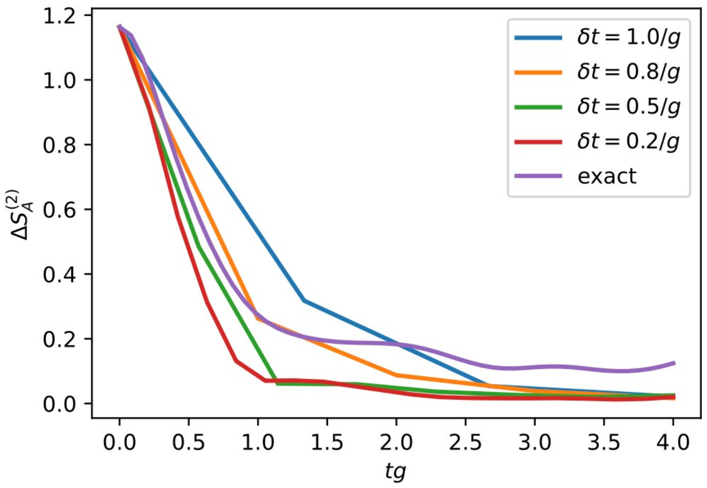
$$H = (H_{ZZ} + H_Z) + \underline{H_{\pm,\mathrm{even}} + H_{\pm,\mathrm{odd}}}$$

(even sites) + (odd sites)



$U(1)$ broken decomposition

$$H = (H_{ZZ} + H_Z) + H_{XX} + H_{YY}$$



$U(1)$ symmetric decomposition has small error than that of $U(1)$ broken decomposition.